1 Simultaneous Equations

1.1 Solving Simultaneous Equations involving one Linear and one Non-Linear Equation in Two Unknown Variables

When solving simultaneous equations in two unknown variables involving one linear and one non-linear equation, we use the **method of substitution**.

The steps are as follows.

Step 1: Rearrange the linear equation to get $x = ...$ or $y = ...$.
Step 2: Substitute the resulting equation in Step 1 into the non-linear equation to form another equation with one variable, then solve for the variable.
Step 3: Substitute back the value obtained in Step 2 into the linear equation and solve for the other variable.
Step 4: Check your solutions in both of the original equations.

Example 1

Solve the simultaneous equations

$$2x + y = 1,$$
$$4x^2 + y^2 = 25.$$

Analytical Thinking
To solve the simultaneous equations in two unknowns x and y, means finding the values of x and y that satisfy both equations.

Solution

Let $\quad 2x + y = 1$................................(1)
and $\quad 4x^2 + y^2 = 25$............................. (2)
From (1):
$\quad y = 1 - 2x$.........................(3)

Step 1
First look for a way to rearrange the linear equation to get $x = ...$ or $y = ...$. Here, $y = 1 - 2x$.

Substitutie (3) into (2):

$$4x^2+(1-2x)^2=25$$
$$4x^2+1-4x+4x^2=25$$
$$8x^2-4x+1=25$$
$$8x^2-4x-24=0$$
$$2x^2-x-6=0$$
$$(2x+3)(x-2)=0$$
$$x=-\frac{3}{2} \quad \text{or} \quad x=2$$

Substitute $x=-\frac{3}{2}$ into (3):

$$y=1-2\left(-\frac{3}{2}\right)$$
$$=4$$

Substitute $x=2$ into (3):

$$y=1-2(2)$$
$$=-3$$

$\therefore\ x=2,\ y=-3$ or $x=-\frac{3}{2},\ y=4.$

Step 2
Substitute the resulting equation in Step 1 into the non-linear equation.

Observe that after the substitution, you will obtain an equation (usually quadratic) which contains only one variable.

Expand and simplify.

It is always easier to solve an equation in its simplified form.

Step 3
After finding the value of one unknown, substituting it into the linear equation to obtain the value of the other unknown.

Step 4
It is always advisable to substitute the solutions obtained into the original equations to double check your answer.

1.2 Finding the Points of Intersection between a Line and a Curve

We can find the points of intersection of two graphs by solving both equations simultaneously.

Example 2

Find the coordinates of the points of intersection of the graphs whose equations are $x-y+6=0$ and $x^2+y^2-6x-8y=0$.

Solution

Let $x-y+6=0$.. (1)
and $x^2+y^2-6x-8y=0$........................... (2)
From (1):
$y=x+6$.. (3)

Step 1
Rearrange the linear equation to get $x=\ldots$ or $y=\ldots$. Here, $y=x+6$.

Substitute (3) into (2):

$$x^2 + (x+6)^2 - 6x - 8(x+6) = 0$$
$$2x^2 - 2x - 12 = 0$$
$$2(x+2)(x-3) = 0$$
$$x = -2 \quad \text{or} \quad x = 3$$

Substitute $x = -2$ into (3):

$$y = -2 + 6$$
$$= -4$$

Substitute $x = 3$ into (3):

$$y = 3 + 6$$
$$= 9$$

$\therefore$ the coordinates of the points of intersection are $(-2,\ 4)$ and $(3,\ 9)$.

Step 2
Substitute the resulting equation in Step 1 into the non-linear equation.

Step 3
After finding the value of one unknown, the value of the other can be obtained by substituting the known value into the linear equation.

Express your answers in coordinates form. It is not correct to write the coordinates as $x = -2$, $y = 4$ or $x = 3$, $y = 9$.

As there are two solution pairs, it means the graph of the straight line intersects the graph of the curve at two points.

1.3 Solving word Problems using Simultaneous Equations

The principle of solving simultaneous equations can also be applied to word problems. The steps to solve such problems can be summarised as follows.

Step 1: Analyse the facts in the problem.
Step 2: Use variables to represent the unknown quantities.
Step 3: Formulate two equations using these variables.
Step 4: Solve the equations simultaneously.

Example 3

The perimeter of a rectangle is 22 cm and its area is 28 cm^2. Find its length and breadth, where the length is longer the breadth.

Step 1
Look for information to form two equations. (Here, we use the perimeter of the rectangle and its area to form two equations.)

Solution

Let the length and breadth of the rectangle be x cm and y cm respectively.
Given that the perimeter is 22 cm,

$$\therefore \quad 2x+2y=22$$
$$x+y=11$$
$$y=11-x \text{..........................(1)}$$

Given that the area is 28 cm^2,

$$\therefore \quad xy=28 \text{...............................(2)}$$

Substitute (1) into (2):

$$x(11-x)=28$$
$$x^2-11x+28=0$$
$$(x-4)(x-7)=0$$
$$x=4 \text{ or } x=7$$

Substitute $x=4$ into (1):

$$y=11-4$$
$$=7$$

Substitute $x=7$ into (1):

$$y=11-7$$
$$=4$$

$\therefore$ the length of the rectangle is 7 cm and the breadth is 4 cm.

Step 2
Use variables to represent the unknown quantities.

Step 3
Formulate two equations using these variables.

Step 4
Solve the equation.

Since the length is longer than the breadth, we reject the other pair of solution, i.e. $x=4,\ y=7$.

Exercise 1

A Solving Simultaneous Equations

1. Solve each of the following simultaneous equations.

(a) $y=x+1$,
$y=x^2-3x-4$.

(b) $24y-1=x$,
$\dfrac{3}{xy}=1$.

(c) $2x+3y=6$,
$(2x+1)^2+6(y-2)^2=49$.

(d) $x+2y=3$,
$x^2+2x-4y^2=7$.

(e) $5x^2-xy=-4$,
$y-4x=-4$.

(f) $y-3x+2=0$,
$2x^2+2y^2+11x+y-66=0$.

2. Solve the simultaneous equations

$$x^2+3y^2=110,$$
$$x+y=12.$$

Give your answer in exact form.

3. Solve the following equation

$$5x-y=2x^2+3y^2=3x+2y-1.$$

4. Find the coordinates of the points of intersection of the line $3x-2y=5$ and the curve $\dfrac{3y}{x}-\dfrac{2x}{y}=1$.

5. If $x = 1$ and $y = -2$ is a solution to the simultaneous equations $ax - y = b$ and $ax^2 + bxy = 3y$, find
   (a) the values of a and b,
   (b) the other solution.

6. The graphs of $5x^2 - 2y^2 = 18$ and $5x + 2a^2y = 18$ intersect at the point $(a, 1)$. Find
   (a) the value of a,
   (b) the coordinates of the other point of intersection.

7. Given that (0, 5), (1, 4) and (2, 7) are the solutions of the equation $y = ax^2 + bx + c$, find the values of a, b and c. Hence, find the value of y when $x = 5$.

8. The line $y = ax - 1$ and the curve $y = x^2 + bx - 5$ intersect at the points $P(4, -5)$ and Q. Find the values of a and b and the coordinates of Q.

B Solving Word Problems involving Simultaneous Equations

9. The sum of two numbers x and y is 10 and the sum of their squares is 350. Given that $y > x$, find the exact values of x and y.

10. A piece of wire 60 cm long is divided into two parts. If each part is made into a square, the total area of the two squares will be 125 cm^2. By setting up two simultaneous equations, find the length of each part.

11. This year, the sum of thrice of John's age and five times that of Peter's is 110, while the sum of squares of their ages is 386. If John is younger than Peter, find John's age in five years' time.

12. An extending ladder has two positions. In position A the length of the ladder is x metres and, when the foot of the ladder is placed 2 metres from the base of a vertical wall, the ladder reaches y metres up the wall.
    In position B the ladder is extended by 0.95 metres and it reaches an extra 1.05 metres up the wall. The foot of the ladder remains 2 m from the base of the wall.
    (a) Use Pythagoras' Theorem for position A and position B to write down two equations in x and y.
    (b) Hence show that $2.1y = 1.9x - 0.2$.
    (c) Using these equations, form a quadratic equation in x. Hence find the values of x and y.

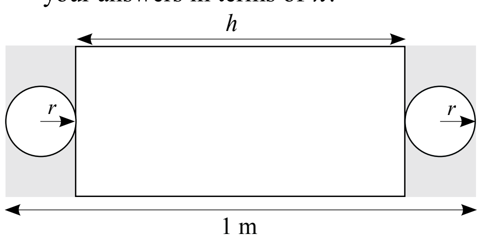
13. The two shorter sides of a right-angled triangle are $(2x - y)$ cm and $(3x + y)$ cm, where $(3x + y) > (2x - y)$. Given that the length of the hypotenuse is $\sqrt{545}$ cm and the difference between the two shorter sides is 1 cm, find the area of the triangle.

14. Two circular discs have a combined surface area of $\frac{29\pi}{2}$ m^2. The sum of the circumferences of the two discs is 10π m. Determine the radius of each disc.

15. A positive whole number has two digits. When the two digits are reversed, a new number is formed. The difference between the squares of the two numbers is 2376. The sum of the two numbers is 66 times the difference between the digits of the original number. Find the two numbers.

16. A rectangular box has a square base. The sum of the lengths of the twelve edges is 40 cm and the sum of the areas of the six faces is 66 cm^2. Find the possible volumes of the box.

17. The diagram shows the net of a cylindrical container of radius r cm and height h cm, where $h > r$. The full width of the metal sheet from which the container is made is 1 m, and the shaded area is waste. The surface area of the container is 1400π cm^2.
    (a) Write down a pair of simultaneous equations for r and h.
    (b) Find the volume of the container, giving your answers in terms of π.



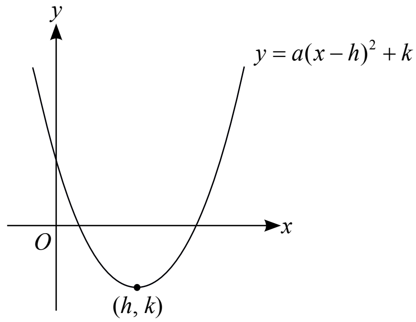
2 Quadratic Equations and Inequalities

2.1 Quadratic Graphs of the Form $y = a(x-h)^2 + k$

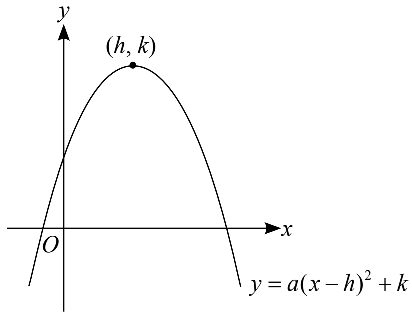
The equation of the form $y = ax^2 + bx + c$, where $a \neq 0$, is called a quadratic function.

We can transform the expression $ax^2 + bx + c$ into the form $a(x-h)^2 + k$ and use it to sketch the corresponding graph of $y = a(x-h)^2 + k$.

The features of the graph of $y = a(x-h)^2 + k$, where $a \neq 0$, are as follows.



$a > 0$



$a < 0$

The shape of the graph depends on the value of a.
If $a > 0$, (h, k) is a minimum point and k is the minimum value of y.
If $a < 0$, (h, k) is a maximum point and k is the maximum value of y.
The curve is symmetrical about the line $x = h$.

Example 1

(a) Find the minimum value of $2x^2 - 8x + 9$ and the corresponding value of x.
(b) Hence, sketch the graph of $y = 2x^2 - 8x + 9$.

Solution

(a) $2x^2 - 8x + 9$

$= 2\left(x^2 - 4x + \frac{9}{2}\right)$

$= 2\left(x^2 - 4x + \mathbf{(-2)^2} - \mathbf{(-2)^2} + \frac{9}{2}\right)$

$= 2\left[(x-2)^2 - (-2)^2 + \frac{9}{2}\right]$

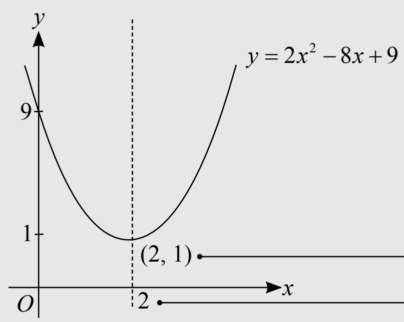
$= 2\left[(x-2)^2 + \frac{1}{2}\right]$

$= 2(x-2)^2 + 1$

Since $(x-2)^2 \geq 0$ for all values of x, then $2(x-2)^2 + 1 \geq 1$ for all values of x.
Hence, the minimum value of y is $\underline{\underline{1}}$ and it occurs when $2(x-2)^2 = 0$, i.e. $x = \underline{\underline{2}}$.

(b) Let $y = 2x^2 - 8x + 9$.

At the y-axis, $x = 0$,
i.e. $y = 2(0)^2 - 8(0) + 9$
$y = 9$



Express $2x^2 - 8x + 9$ as $a(x-h)^2 + k$ by completing the square.

Take 2 out from the expression so that the coefficient of x^2 is 1.

Add and subtract $(\frac{1}{2} \times \text{coefficient of } x)^2$, i.e. $\left[-\frac{1}{2} \times (-4)\right] = (-2)^2$.

Recall $(a-b)^2 = a^2 - 2ab + b^2$. In this case, $(x-2)^2 = x^2 - 4x + 2^2$.

Multiply 2 into the expression.

The procedure of sketching a quadratic graph can be summarised as follows.

(i) Determine the shape of the graph by identifying whether the coefficient of x^2 is positive or negative. In this case, since $a > 0$, the graph is $\bigcup$ shaped.

(ii) Find the axial intercept.

(iii) Use completing the square method to obtain the turning point of the graph.

State the minimum point and the y-intercept.

Note that the parabola is symmetrical in the line $x = 2$.

The sketch the graph of $y = 2x^2 - 8x + 9$ is as shown.

2.2 Discriminant of Quadratic Equations

An equation of the form $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$, is called a quadratic equation.

The roots of the equation can be found by using the quadratic formula which is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Note that $D = b^2 - 4ac$ is called the discriminant of the quadratic equation.

The discriminant, $D = b^2 - 4ac$ determines the nature of the two roots.

$D > 0$	two real and distinct (or unequal) roots.
$D = 0$	two real and equal (or repeated) roots.
$D < 0$	no real roots.
$D \geq 0$	real roots.

Using the Discriminant to Identify the types of Roots of a Quadratic Equation

Example 2

Show that the quadratic equation $x^2 + 2x + 1 = 0$ has equal roots.

Solution

Let $a = 1$, $b = 2$ and $c = 1$.

Discriminant, D

$= b^2 - 4ac$

$= (2)^2 - 4(1)(1)$

$= 0$

Begin by listing the coefficients of the quadratic equation.

Next, substitute the values of a, b and c into D.

Since the value obtained is 0, we say that the equation has equal roots.

If the value obtained is more than 0, we say that the equation has real and distinct roots.

If the value obtained is less than 0, we say that the equation has no real roots.

Using the Discriminant to prove the nature of a Quadratic Equation

Example 3

Prove that the equation $x^2-(k+1)x=2-k$ has distinct roots for any real values of k.

Solution

$$x^2-(k+1)x=2-k$$
$$x^2-(k+1)x+(k-2)=0.$$

Let $a=1,\ b=-(k+1)$ and $c=k-2.$

Discriminant, D

$=b^2-4ac$

$=[-(k+1)]^2-4(1)(k-2)$

$=k^2+2k+1-4k+8$

$=k^2-2k+9$

$=(k-1)^2+8$

Since $(k-1)^2\geq 0$ for all real values of k, the expression $(k-1)^2+8$ is always positive.

$\therefore$ the given equation has distinct roots for any real values of k.

Always arrange the equation in the form $ax^2+bx+c=0$.

List the coefficients of the quadratic equation.

Substitute the values of $a,\ b$ and c into D.

Expand the expression.

Complete the square for the expression k^2-2k+9.

Using the Discriminant to find the values of the Unknown Constants of a Quadratic Equation

The following table shows the relations among the discriminant of the quadratic equation $ax^2+bx+c=0$, where $a\neq 0$, and the nature of its roots.

Discriminant, $D=b^2-4ac$	**Nature of Roots of $ax^2+bx+c=0$**
$D>0$	• real and distinct roots • real and different roots • unequal roots • distinct roots • one positive and one negative root
$D=0$	• real and equal roots • repeated roots • coincident roots
$D<0$	• no real roots • complex roots
$D\geq 0$	• real roots

Chapter 2.indd-id5.indd 9 2015-11-17 17:01:08

Example 4

Find the value of k for which the quadratic equation $x^2+3x+k=0$ has equal roots.

Solution

Let $a=1$, $b=3$ and $c=k$.

For equal roots, $b^2-4ac=0$.

So $(3)^2-4(1)(k)=0$

$9-4k=0$

$4k=9$

$\therefore\ k=\dfrac{9}{4}$.

Refer to the roots mentioned in the question.
(Here, equal roots $\Rightarrow$ $b^2-4ac=0$.)

List the coefficients of the quadratic equation.

Substitute the values of a, b and c into $b^2-4ac=0$.

Solve for k.

Example 5

Given the quadratic equation $kx^2=6x-3$, where k is a non-zero constant, find the range of values of k for which the equation has

(a) real and distinct roots,

(b) real roots.

Refer to the roots mentioned in the question.
(a) Real and distinct roots $\Rightarrow b^2-4ac>0$.)
(b) Real roots $\Rightarrow$ $b^2-4ac\ge 0$.)

Solution

$kx^2=6x-3$

$kx^2-6x+3=0$

Let $a=k$, $b=-6$ and $c=3$.

(a) For real and distinct roots, $b^2-4ac>0$.

So $(-6)^2-4(k)(3)>0$

$36-12k>0$

$12k-36<0$

$12k<36$

$\therefore\ k<3$ except 0.

(b) For real roots, $b^2-4ac\ge 0$.

So $(-6)^2-4(k)(3)\ge 0$

$\therefore\ k\le 3$, except 0.

Express the equation in the form $ax^2+bx+c=0$.

Multiply (–) sign on both sides. Observe that you would need to reverse the inequality sign.

Common Mistake
It is not correct to do the following.

$36-12k>0$

$-12k>-36$

$k>3$

Since k is non-zero constant, we reject $k=0$.

2.3 Relationship between a Line and a Cirve

When solving the equations of the line and curve simultaneously gives a quadratic equation, the nature of the roots of that equation can be summarised in the following table.

Types of Intersection	**Discriminant**
When the line intersects the curve at two distinct points, then	$D > 0$
When the line is a tangent to the curve or just cuts the curve at one point, then	$D = 0$
When the line does not intersect the curve, then	$D < 0$
When the line intersects the curve, then	$D \geq 0$

Using the Discriminant to find the values of the unknown constant involving a Line and a Curve

Follow these steps to find the values of the unknown constant involving a line and a curve.

Consider a curve with equation $y = \mathrm{g}(x)$ and a line with equation $y = mx + k$.

Step 1 : Solve the curve and line simultaneously to form a quadratic equation $ax^2 + bx + c = 0$.

Step 2 : Check the relation between the line and the curve.

Step 3 : Use the discriminant to solve the resulting equation in Step 1.

Example 6

Find the range of values of m for which the line $y - x = -2m$ and the curve $y = x(x-2)$ do not intersect.

Solution

Let $y - x = -2m$ .... (1)

and $y = x(x-2)$ .... (2)

From (1):

$y = x - 2m$ .... (3)

Substitute (3) into (2):

$x - 2m = x(x-2)$

$x^2 - 3x + 2m = 0$ .... (4)

First rearrange the linear equation to make y in terms of x.

Solve simultaneously to form a quadratic equation $ax^2 + bx + c = 0$.

Let $a = 1$, $b = -3$ and $c = 2m$.

For no real roots, $b^2 - 4ac < 0$.

$\therefore \quad (-3)^2 - 4(1)(2m) < 0$

$9 - 8m < 0$

$8m - 9 > 0$

$\therefore \; m > \dfrac{9}{8}$

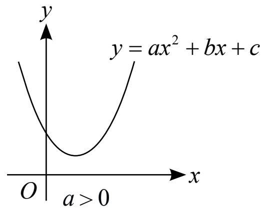
Check the relation between the line and the curve.
(Here, the line and the curve do not intersect. means that it has no real roots, i.e. $b^2 - 4ac < 0$)

Substitute the values of a, b and c into the discriminant.

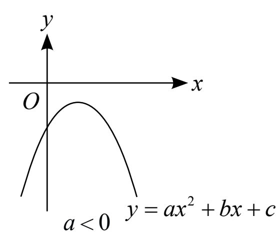
2.4 Interpret the Conditions of $ax^2 + bx + c$

Consider the general form of a quadratic expression $ax^2 + bx + c$, where a, b and c are real and $a \neq 0$.

If the quadratic expression **is always positive** for all values of x, then its corresponding graph is always *above* the x-axis. Hence, there is no intersection between the curve and the x-axis. So, the discriminant, $D < 0$. (Refer to the diagram on the right)



If the quadratic expression **is always negative** for all values of x, then its corresponding graph is always *below* the x-axis. Hence, there is no intersection between the curve and the x-axis. So, the discriminant, $D < 0$. (Refer to the diagram on the right)



The table below summarises four different conditions of the expression $ax^2 + bx + c$.

When the expression $ax^2 + bx + c$	Coefficient of x^2	Discriminant
is always positive, then	$a > 0$	$D < 0$
is always negative, then	$a < 0$	$D < 0$
is never positive, then	$a < 0$	$D \leq 0$
is never negative, then	$a > 0$	$D \leq 0$

Example 7

Find the range of values of k for which $kx^2 + 2x + 1$ is always positive for all real values of x.

Solution

$a > 0$ and Discriminant < 0

i.e. $k > 0$ $\quad$ $b^2 - 4ac < 0$

$$(2)^2 - 4(k)(1) < 0$$

$$4 - 4k < 0$$

$$4k - 4 > 0$$

$\therefore$ $\underline{\underline{k > 1}}$

First check the condition of the expression $ax^2 + bx + c$. Here it states that the expression is **always positive**.

Refer to the table in the Section 2.4 and write down the condition.

Example 8

Find the range of values of m for which $mx^2 + 2x - 3$ is never positive for all real values of x.

Solution

$a < 0$ and Discriminant ≤ 0

i.e. $m < 0$ $\quad$ $(2)^2 - 4(m)(-3) \leq 0$

$$12m + 4 \leq 0$$

$$m \leq -\frac{1}{3}$$

$\therefore$ $\underline{\underline{m \leq -\frac{1}{3}}}$

First check the condition of the expression $ax^2 + bx + c$. Here it states that the expression is **never positive**.

Refer to the table in the Section 2.4 and write down the condition.

2.5 Sum and Product of Roots of a Quadratic Equation

If α and β are the two roots of the equation $ax^2 + bx + c = 0$, where a, b and c are real numbers with $a \neq 0$, then

$$\text{Sum of Roots} = \alpha + \beta = -\frac{b}{a}$$

$$\text{Product of Roots} = \alpha\beta = \frac{c}{a}$$

Some useful identities:

(a) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

(b) $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)$

(c) $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

(d) $\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$

(e) $\alpha^3 - \beta^3 = (\alpha - \beta)(\alpha^2 + \alpha\beta + \beta^2)$

The following steps may serve as a guide in finding the values of the expression relating the roots of the quadratic equation.

Step 1: Find the values of $\alpha + \beta$ and $\alpha\beta$ of the original equation.

Step 2: Rewrite the required expressions in terms of $\alpha + \beta$ and of $\alpha\beta$.

Step 3: Substitute the value of $\alpha + \beta$ and of $\alpha\beta$ into the required expressions.

Example 9

If α and β are the roots of the equation $2x^2 + 6x + 3 = 0$, where $\alpha < \beta$, find the values of the following expressions.

(a) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ **(b)** $\alpha^2 + \beta^2$ **(c)** $\alpha - \beta$

Solution

Given $2x^2 + 6x + 3 = 0$, let $a = 2$, $b = 6$ and $c = 3$.

List the coefficients of the quadratic equation.

Sum of roots

$= \alpha + \beta$

$= -\dfrac{b}{a}$

$= -\dfrac{6}{2}$

$= -3$

Product of roots

$= \alpha\beta$

$= ac$

$= \dfrac{3}{2}$

Begin by finding the values of $\alpha + \beta$ and $\alpha\beta$ of the original equation.

(a) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$

Rewrite the expression as a single fraction.

$= \dfrac{\alpha + \beta}{\alpha\beta}$

Substitute the values of $\alpha+\beta$ and $\alpha\beta$ into the expression.

$= \dfrac{-3}{\frac{3}{2}}$

$= \underline{\underline{-2}}$

(b) $\alpha^2 + \beta^2$

$= (\alpha + \beta)^2 - 2\alpha\beta$

$= (-3)^2 - 2\left(\frac{3}{2}\right)$

$= \underline{\underline{6}}$

Refer to the useful identities on the previous page.

Substitute the values of $\alpha+\beta$ and $\alpha\beta$ found into the expression.

(c) $(\alpha - \beta)^2$

$= \alpha^2 - 2\alpha\beta + \beta^2$

$= \alpha^2 + \beta^2 - 2\alpha\beta$

$= 6 - 2\left(\frac{3}{2}\right)$

$= 3$

$\therefore \quad \alpha - \beta = \underline{\underline{\pm\sqrt{3}}}$

Expand and simplify.

Use the result obtained in **(b)**.

Example 10

Given that α and β are the roots of the equation $x^2 + (2k+1)x + (k^2 - 2) = 0$. If $\alpha^2 + \beta^2 = 3\alpha\beta - 10$, find the values of k.

Solution

Let $a = 1$, $b = (2k+1)$ and $c = (k^2 - 2)$.

Sum of roots $= \alpha + \beta = -(2k+1)$

Product of roots $= \alpha\beta = k^2 - 2$

Given $\alpha^2 + \beta^2 = 3\alpha\beta - 10$

$(\alpha + \beta)^2 - 2\alpha\beta = 3\alpha\beta - 10$

$(\alpha + \beta)^2 = 5\alpha\beta - 10$

$[-2(k+1)]^2 = 5(k^2 - 2) - 10$

$4k^2 + 4k + 1 = 5k^2 - 20$

$k^2 - 4k - 21 = 0$

$(k+3)(k-7) = 0$

$k = \underline{\underline{-3 \text{ or } 7}}$

Begin by finding the values of $\alpha + \beta$ and $\alpha\beta$ of the original equation.

Change $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$.

Substitute the values of $\alpha+\beta$ and $\alpha\beta$ into the equation.

Expand and simplify.

2.6 Forming a Quadratic Equation from its Roots

The steps in forming the new quadratic equation from its roots are as follows.

Step 1 : Find the values of the sum and the product of roots of the original equation.
Step 2 : Find the values of the sum and the product of roots of the new equation.
Step 3 : Write the new quadratic equation as

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

Example 11

Given that the roots of the equation $2x^2 - x - 7 = 0$ are α and β, find the quadratic equation in x whose roots are $\alpha - \frac{2}{\beta}$ and $\beta - \frac{2}{\alpha}$.

Solution

Sum of roots $= \alpha + \beta = \frac{1}{2}$

Product of roots $= \alpha\beta = -\frac{7}{2}$

Sum of roots of the new quadratic equation

$$\left(\alpha - \frac{2}{\beta}\right) + \left(\beta - \frac{2}{\alpha}\right)$$

$$= \alpha + \beta - \frac{2}{\beta} - \frac{2}{\alpha}$$

$$= \alpha + \beta - \frac{2(\alpha + \beta)}{\alpha\beta}$$

$$= \frac{1}{2} - \frac{2\left(\frac{1}{2}\right)}{-\frac{7}{2}}$$

$$= \frac{11}{14}$$

Step 1
Begin by finding the values of the sum and the product of the roots of the original equation.

Step 2
Find the values of the sum and the product of roots of the new equation.

Add the two expressions.

Substitute the values of $\alpha+\beta$ and $\alpha\beta$ into the expression.

Product of roots of the new quadratic equation

$$= \left(\alpha - \frac{2}{\beta}\right)\left(\beta - \frac{2}{\alpha}\right)$$

$$= \alpha\beta + \frac{4}{\alpha\beta} - 4$$

$$= -\frac{7}{2} + \frac{4}{-\frac{7}{2}} - 4$$

$$= -\frac{121}{14}$$

The new quadratic equation is

$$x^2 - \left(\frac{11}{14}\right)x + \left(-\frac{121}{14}\right) = 0$$

$$\underline{\underline{14x^2 - 11x - 121 = 0}}$$

Multiply out the brackets.

Substitute the values of $\alpha + \beta$ and $\alpha\beta$ into the expression.

Recall
The quadratic equation is
$x^2 -$ (sum of roots)$x +$ (product of roots) $= 0$

Step 3
Write down the new equation.

Mutiply 14 on both sides of the equation.

2.7 Quadratic Inequalities

An inequality that involves quadratic expressions in one variable is called a **quadratic inequality**. For example, $x^2 - 3x + 2 < 0$, $3x(x - 2x) \geq 8$ and $(x - 1)(2x + 3) > 0$ are known as quadratic inequalities in x.

To solve a quadratic inequality graphically, follow these steps.

Step 1: Rewrite the inequality such that its left side is in the form $ax^2 + bx + c$, where $a > 0$, and its right side is zero. For example, $x^2 - 3x + 4 > 0$.

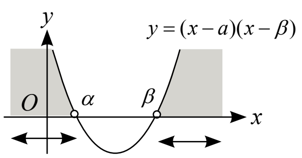
Step 2: Find the roots of $ax^2 + bx + c = 0$, i.e. the values of x. Here, we assume $x = \alpha$ or $x = \beta$, where $(\beta > \alpha)$

Step 3: Sketch the quadratic graph, indicating the two roots.

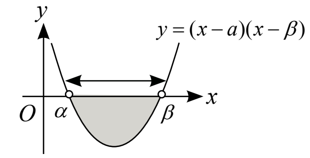
Step 4: Identify the range of values of x that satisfies the inequality.

(i) <u>if the inequality sign is more than or more than or equal to</u> ($>$ or $\geq$), then the range of x is $x > \beta$ or $x < \alpha$).
Refer to the diagram **(a)**, we see that the range of value of x is *above* the x-axis.

(ii) <u>if the inequality sign is less than or less than or equal to</u> ($<$ or $\leq$), then the range of x is $\alpha < x < \beta$).
Refer to the diagram **(a)**, we see that the range of value of x is *below* the x-axis.



(a)



(b)

Example 12

Solve the inequality $x^2 + x > 2$.

Solution

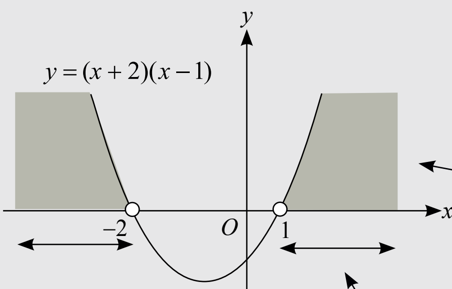
$x^2 + x > 2$

$x^2 + x - 2 > 0$

$(x+2)(x-1) > 0$

Let $(x+2)(x-1) = 0$

$x = -2$ or $x = 1$



$\therefore$ $\underline{\underline{x < -2 \text{ or } x > 1}}$.

Step 1: Arrange the expression in the form $ax^2 + bx + c$.

Factorise the expression.

Step 2: Find the roots of $ax^2 + bx + c = 0$, i.e. the values of x.

Step 3: Sketch the graph of $y = (x+2)(x-1)$, indicating the two roots.

Remark
Your sketch of the graph does not need to be accurate. All you need to know is that the graph is '$\bigcup$- shaped' and crosses the x-axis at -2 and -1.

Step 4: Identify the range of values of x that satisfies the inequality.

Since the inequality sign is <u>more than 0</u>, i.e. $(x+2)(x-1) > 0$, the range of values of x refers to the part of the curve *above* the x-axis.

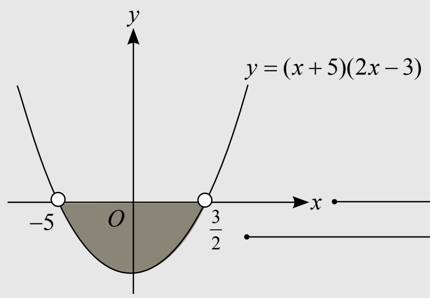
Example 13

Find the range of values of x for which $2x^2+7x<15$.

Solution

$$2x^2+7x<15$$
$$2x^2+7x-15<0$$
$$(x+5)(2x-3)<0$$
Let $(x+5)(2x-3)=0$

$\therefore$ the range of values of x is $\underline{\underline{-5\le x<\frac{3}{2}}}$.



$\therefore$ $\underline{\underline{-5<x<\frac{3}{2}}}$

Arrange the expression in the form ax^2+bx+c.

Factorise the quadratic expression.

Misconception
It is wrong to write something like $x<-5$ or $x<\frac{3}{2}$ because x may not be less than -5.

Also, $-5>x>\frac{3}{2}$ is incorrect because x cannot be less than -5 and greater than $\frac{3}{2}$ at the same time.

Sketch of the graph of $y=(x+5)(2x-3)$.

Since the inequality sign is <u>less than</u> ($<$), i.e. $2x^2+7x-15<0$, the range of values of x refer to the part of the graph *below* the x-axis.

2.8 Solving Compound Inequalities

Example 14

Find the range of values of x for which $12+4x>x^2$ and $5x-3\geq 2$.

Solution

Consider $12+4x>x^2$

$12+4x-x^2>0$

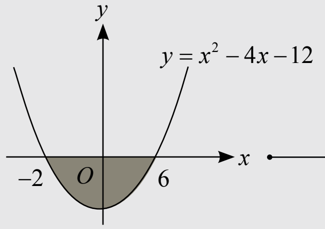
$x^2-4x-12<0$

Multiply $(-)$ sign on both sides and reverse the inequality sign.

Let $x^2-4x-12=0$

$(x+2)(x-6)=0$

$x=-2$ or $x=6$



Sketch the graph of $y=x^2-4x-12$. For $x^2-4x-12<0$ $(y<0)$, it refers to the part of the graph below the x-axis.

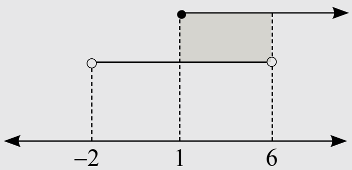
$\therefore\ -2<x<6$

Next, consider $5x-3\geq 2$

$5x\geq 5$

$x\geq 1$

Draw a number line to illustrate the two inequalities.



The overlapping lines refer to the range of values of x that satisfies both inequalities.

$\therefore$ the range of values of x is $\underline{\underline{1\leq x<6}}$.

2.9 Solving Inequalities involving the Discriminant

Example 15

Find the range of values of p for which the equation $3x^2 + 3px + p^2 = 1$ has two distinct real roots.

Solution

Rewriting the equation gives $3x^2 + 3px + p^2 - 1 = 0$.
Let $a = 3,\ b = 3p$ and $c = p^2 - 1$

$$b^2 - 4ac > 0$$
$$\therefore\ (3p)^2 - 4(3)(p^2 - 1) > 0$$
$$9p^2 - 12p^2 + 12 > 0$$
$$-3p^2 + 12 > 0$$
$$p^2 - 4 < 0$$
$$(p+2)(p-2) < 0$$

$\therefore$ the range of values of p is $\underline{\underline{-2 < p < 2}}$.

For two distinct real roots, it means $b^2 - 4ac > 0$.

Substitute the values of a, b and c into $b^2 - 4ac$.

Expand the left side of the inequality which leads to the quadratic form $ax^2 + bx + c > 0$.

Factorise the LHS of the inequality.

Example 16

Find the range of values of m such that the line $y = mx - 1$ meets the curve $y = 3x - 4 - 3x^2$.

Solution

Let $\quad y = mx - 1$....................(1)

and $\quad y = 3x - 4 - 3x^2$...........(2)

Equating (1) and (2):

$$mx - 1 = 3x - 4 - 3x^2$$
$$3x^2 + mx - 3x - 1 + 4 = 0$$
$$3x^2 + (m-3)x + 3 = 0$$

When the line meets the curve, it means that $b^2 - 4ac \geq 0$.

Solve simultaneously to eliminate y.

Arrange the equation in the form $ax^2 + bx + c = 0$.

Let $a = 3,\ b = m - 3$ and $c = 3$.

For real roots, $b^2 - 4ac \geq 0$

$\therefore \quad (m-3)^2 - 4(3)(3) \geq 0$

$m^2 - 6m + 9 - 36 \geq 0$

$m^2 - 6m - 27 \geq 0$

$(m+3)(m-9) \geq 0$

$m \leq -3 \quad$ or $\quad m \geq 9$

$\therefore$ the range of values of m is $\underline{\underline{m \leq -3 \text{ or } m \geq 9}}$.

Substitute the values of $a,\ b$ and c into $b^2 - 4ac \geq 0$.

Expand the left side of the inequality which leads to the quadratic form $ax^2 + bx + c \geq 0$.

Factorise the inequality.

Exercise 2

A Quadratic Graphs of the Form $y = a(x-h)^2 + k$

1. **(a)** By expressing $y = 2x^2 - 4x + 5$ in the form $a(x-h)^2 + k$, where $a,\ h$ and k are constants, state the minimum value of y and the corresponding value of x for which this occurs.

(b) Hence sketch the graph of $y = 2x^2 - 4x + 5$, indicating clearly the coordinates of the turning point and the y-intercept.

2. It is given that $4 + 3x - x^2 = a(x-b)^2 + c$ for all values of x.

(a) Find the values of $a,\ b$ and c.

(b) Hence, find

(i) the maximum value of $4 + 3x - x^2$, and

(ii) the corresponding value of x.

(c) Sketch the curve $y = 4 + 3x - x^2$.

3. Given that $7 - x - 3x^2 = a - b(x+c)^2$ for all real values of x, find the values of $a,\ b$ and c. Hence, state

(a) the maximum value of $7 - x - 3x^2$,

(b) the value of x at which the maximum value occurs.

Sketch the curve $y = 7 - x - 3x^2$.

B Nature of Roots of Quadratic Equations

4. Find the discriminant and hence determine the nature of roots for each of the following equations.

(a) $5x + 4x + 6 = 0$ **(b)** $x^2 = 4(2x - 4)$

(c) $(2x-1)(x+3) = 5$

5. Find the values of k in which $2x^2 + 4x + (k-2) = 0$ has real and distinct roots.

6. Determine the range of k such that $x^2 - 2x + k(k - 2x) + 3 = 0$ has real roots.

7. Find the values of k in which the equation $(x-3)(x+k) = k(x+2)$ has two equal roots.

8. Find the range of values of a in which the quadratic equation $3ax^2 - 5x = 4(x-1)$ has no real roots.

9. Find the real values of k such that the equation $kx^2 + (2k-1)x + k - 2 = 0$, where $k \neq 0$, has one positive and one negative root.

10. Show that the roots of the equation $x^2 + 2ax + b(x + 2a) = 0$ are real for any real values of a and b. Hence, write down a relation between a and b if the roots are real and equal.

11. A quadratic equation is such that $qx^2 + 3px + 9q = 0$, where $q \neq 0$.
   (a) Find the ratio of $p : q$ such that both p and q are positive if the above equation has coincident roots.
   (b) Write an inequality relating p and q if the equation has two unequal real roots.

12. If a is a real number and $a \neq -3$, prove that the equation $(a^2 + 9)x^2 - 2(a - 3)x + 2 = 0$ has no real roots.

13. Prove that the roots of the equation $2x^2 - x + k^2 + 2kx = 0$ are real and distinct if $k < \frac{1}{4}$.

14. Show that the quadratic equation $2k(x + 1) = 3 - 3x^2$ has real roots for all real values of k.

C Relationship between a Line and a Curve

15. Find the values of k for which the line $3x + y = 2k$ and the curve $y = (x - 1)^2 + k$ do not intersect.

16. Find the value of k such that the line $y + 3x = k$ is a tangent to the curve $y = x^2 + 5$.

17. Find the range of values of m in which the line $y = -x + \frac{13}{6}$ meets the curve $xy = m$.

18. The equation of a curve is $y = 3x^2 - kx + 2k - 4$, where k is a constant.
   (a) Show that the line $y = 2x + 5$ intersects the curve for all real values of k.
   (b) If $k = 7$, determine the number of times the curve intersects the x-axis.

19. The line has equation $y = kx + 3$, where k is a constant. The curve has equation $x^2 + 2xy = 7$.
   (a) Show algebraically that the x-coordinates of the points of intersection of the line and the curve satisfy the equation $ax^2 + bx + c = 0$, where a, b and c are real numbers.
   (b) Hence, find the range of values of k if the line meets the curve at two real and different roots.

20. The line $y - mx = 2m$, where m is a positive real number, is a tangent to the curve $y = -x^2 + 2x - 1$. Find
   (a) the value of m,
   (b) the coordinates of their point of contact.

D Conditions for $ax^2 + bx + c$ to be always Positive or Negative

21. Find the range of values of c for which the expression $cx^2 - 3x + 1$ is always positive for all real values of x.

22. Find the range of values of k such that the quadratic expression $-x^2 - kx + 2k + 4$ is always negative for all real values of x.

23. Find the range of values of c for which $3x^2 + 4x + c$ has no negative values.

24. Find the range of values of m such that $2x^2 + x + m > 0$ for all real values of x.

25. Find the range of values of k for which $kx^2 + 2(x - 1) < \frac{k}{4} - 1$, where $k \neq 0$, for all real values of x.

26. Find the range of values of m for which $mx^2 + 2x - 3$ is never positive for all real values of x.

27. Find the range of values of k for which
   (a) the graph of $y = x^2 + 2kx + (k - 1)(k - 3)$ lies entirely above the x-axis,
   (b) the graph of $y = -3x^2 + 6x + 5 - 2k$ lies entirely below the x-axis.

28. Find the value(s) or range of values of k for which the graph of
   (a) $y = 3x^2 + 2x - k$ intersects the x-axis at two distinct points,
   (b) $y = 2x^2 + x + 2k$ does not cut the x-axis,
   (c) $y = 5kx^2 - 6kx + 18$ touches the x-axis at one point.

29. Given that x is real, prove that $-2x^2+12x-19$ is always negative.

30. Show that the roots of the equation $ax^2+(3a+b)x+3b=0$ are real for all values of a and b.

E Relationship between the Roots and Coefficients of a Quadratic Equation

31. Find the sum and product of the roots of the following equations.

(a) $6x^2-4x+3=0$ **(b)** $4x^2+3=0$

(c) $3x^2+3x-2=4x$ **(d)** $3(2x-1)=2(x^2-4)$

32. The equation $2x^2-7x+3=0$ has roots α and β. Find the values of each of the following expressions.

(a) $\alpha^2\beta+\beta^2\alpha$ **(b)** $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$

33. The roots of the quadratic equation $2x^2+4x-5=0$ are α and β. State the value of $\alpha+\beta$ and $\alpha\beta$. Find

(a) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$, **(b)** $(2\alpha+3)(2\beta+3)$,

(c) $\alpha^2+\beta^2$, **(d)** $(\alpha-\beta)^2$.

34. If α and β are the roots of the equation $2x^2+5x-4=0$, find the values of the following expressions.

(a) $\left(\alpha-\dfrac{\alpha}{\beta}\right)\left(\beta-\dfrac{\beta}{\alpha}\right)$ **(b)** $2\alpha^2+2\beta^2$

(c) $\alpha-\beta$ **(d)** $\alpha^3\beta+\alpha\beta^3$

35. (a) Given that the roots of the equation $x^2-(q-5)x=\dfrac{q}{2}$ are α and $\alpha+5$, find the possible values of q.

(b) If the equation $x^2+(2-k)x+k=0$ has positive roots which differ by 2, find the value of each root and of k.

36. The roots of the equation $x^2+ax+b=0$, where a and b are α and β. The roots of the equation $x^2+bx+a=0$ are $\alpha+2$ and $\beta+2$. Show that $a=0$ and hence find the value of b.

37. Given that one root of the equation $x^2+kx-6=0$ is the reciprocal of a root of the equation $2x^2+kx-1=0$, find the positive value of k.

F Forming Quadratic Equations

38. The roots of the equation $2x^2-3x-4=0$ are α and β. Form a quadratic equation whose roots are $\dfrac{2}{\alpha}$ and $\dfrac{2}{\beta}$.

39. The roots of the quadratic equation $4x^2+6x+1=0$ are α and β. Find the quadratic equation whose roots are $\left(\alpha+\dfrac{1}{\alpha}\right)$ and $\left(\beta+\dfrac{1}{\beta}\right)$.

Give your answer in the form $ax^2+bx+c=0$, where a, b and c are integers.

40. (a) Form a quadratic equation for which the sum of roots is 2 and the sum of the squares of the roots is 18.

(b) The roots of the quadratic equation $3x^2-5x+7=0$ are $\alpha+1$ and $\beta+1$.

(i) Find the value of $\alpha+\beta$ and of $\alpha\beta$.

(ii) Find the quadratic equation in x whose roots are $\dfrac{\alpha^2}{2}$ and $\dfrac{\beta^2}{2}$.

41. The roots of the equation $x^2-3x+2=0$ are α and β while the roots of the equation $hx^2-7x+k=0$ are $\alpha+2$ and $\beta+2$.

(a) Calculate the numerical values of h and k.

(b) Form a quadratic equation in x, with integer coefficients whose roots are $\dfrac{\alpha}{\beta}+1$ and $\dfrac{\beta}{\alpha}+1$.

42. The roots of the quadratic equation $kx^2 - 3x - 2 = 0$ are α and β. Given that the product of the roots is $-\frac{1}{2}$, find

(a) the value of k and state the sum of the roots,

(b) the quadratic equation in x whose roots are $(\alpha^2 - 4)$ and $(\beta^2 - 4)$.

G Solving Quadratic Inequalities

43. Solve the following inequalities.

(a) $3x^2 + 4x > 4$

(b) $-4x^2 + 8x + 5 \leq 0$

(c) $(2x - 3)(3x + 1) \geq 4x(2x - 3)$

(d) $-2x(x + 2) > x - 12$

(e) $\dfrac{(3x - 1)(x + 1)}{4} \geq \dfrac{(2x - 3)(x - 1)}{2}$

(f) $2x - 3 < (3x - 2)^2$

44. Find the range of values of x for which $-3x^2 + 2x + 38 < -x - 22$. Hence, solve $(y + 2)^2 - y > 21$.

45. Using a separate diagram for each part, represent on the number line the solution set of

(a) $3(x + 7) > 5(1 - x)$, **(b)** $x < 2(x^2 - 3)$.

State the set of values of x which satisfy both of these inequalities.

46. Find the value of c and of d for which

(a) $\{x : -4 < x < 7\}$ is the solution set of $x^2 + cx < d$,

(b) the set of values of x that satisfies $2x^2 + cx > 20$ is $x < -5$ or $x > d$.

47. A triangle has a base of length $(13 - 2x)$ m and a perpendicular height of x m. Calculate the range of values of x for which the area of the triangle is greater than 3 m^2.

48. Find the range of values of x which satisfies.

(a) $x^2 > \dfrac{9x + 5}{2}$ and $x^2 - 2x \leq 15$

(b) $5x - 6 \leq x^2 < 16$

(c) $4 - 3x < x^2 \leq 36$

49. Given that $-6 < x < 6$, find the range of values of x which satisfy both the inequalities $2x^2 + 3x > 2$ and $3 - 2x < 5$.

H Solving Inequalities involving the Discriminant

50. Find, in exact form, the range of values of k for which the equation $x^2 + (k - 3)x + 2 = 0$ has real roots.

51. Given that the equation $2m^2 + 2m = -x^2 - 2mx + 3$ has no real roots, find the range of values of m.

52. Find the range of values of k for which the curve $y = x^2 - 5x + k$ meets the line $y = kx - 8$ at two distinct points. Hence state the values of k for which the line is a tangent to the curve.

53. Find the range of values of k for which the expression $kx^2 - 2kx + 3k + 1$, where $k \neq 0$,

(a) is never positive,

(b) is always positive,

(c) is always negative,

(d) is never negative,

for all values of x.

54. Find the least value of the integer of a for which $ax^2 - 8x + a > 6$ for all real values of x.

55. Find the largest prime number n such that the expression $2x^2 + nx + 7$ is positive for all values of x.

56. Find the range of values for ab such that the line $x + y = a$ does not intersect the curve $\dfrac{1}{x} + \dfrac{1}{y} = b$.

3 Polynomials

An algebraic expression which makes up of terms each of the form ax^n, where a is a constant and n is a non-negative integer, is called a **polynomial** in x.

Expressions such as $x+1$, $2x^2-5x+4$ and x^3+x^2-5x+3 are known as **polynomials**.

3.1 Finding the Unknown Constants in an Identity

When two polynomials expressions take the same values for every value of the variable, they are said to be an **identity**. For example, $(x+2)^2 = x^2+4x+4$ is an identity.

There are two methods to find the unknown constants in an identity. They are **comparing the coefficients** of the polynomials and the **substitution method**.

Method 1 (Comparing the Coefficients)

Example 1

Given that $A(x-1)^2+B(x-1)+C = 2x^2-x-8$ for all values of x, find the values of A, B and C.

Solution

$$A(x-1)^2+B(x-1)+C = 2x^2-x-8$$

Multiply out left side of the equation.

$$A(x^2-2x+1)+Bx-B+C = 2x^2-x-8$$

Group the terms according to the degree of x.

$$Ax^2-2Ax+A+Bx-B+C = 2x^2-x-8$$

$$Ax^2-2Ax+Bx+A-B+C = 2x^2-x-8 \text{ .... (1)}$$

Equate the coefficients of x^2 in (1):

$$A = 2 \text{ .... (2)}$$

Equate the coefficients of x in (1):

$$-2A+B = -1 \text{ .... (3)}$$

Substitute (2) into (3):

$$\therefore \quad -2(2)+B = -1$$

$$B = 3 \text{ .... (4)}$$

Equate the constant terms in (1):

$$A - B + C = -8 \text{ .... } (5)$$

Substitute (2) and (4) into (5):

$$2 - (3) + C = -8$$
$$C = -7$$

$\therefore$ $\underline{\underline{A = 2, B = 3 \text{ and } C = -7.}}$

Method 2 (Substitution Method)

Example 2

Given that $3x^2 - 5x + 1 = A(x-1)(x-2) + B(x-1) + C$ for all values of x, find the values of A, B and C.

Solution

Let $3x^2 - 5x + 1 = A(x-1)(x-2) + B(x-1) + C$ .... (1)

Substitute $x = 1$ into (1):

$$3(1)^2 - 5(1) + 1 = A(1-1)(1-2) + B(1-1) + C$$
$$C = -1$$

Substitute $x = 2$ into (1):

$$3(2)^2 - 5(2) + 1 = A(2-1)(2-2) + B(2-1) + C$$
$$3 = B + C$$
$$3 = B + (-1)$$
$$\therefore \quad B = 4$$

Substitute $x = 0$ into (1):

$$3(0)^2 - 5(0) + 1 = A(0-1)(0-2) + B(0-1) + C$$
$$1 = 2A - B + C$$
$$1 = 2A - (4) + (-1)$$
$$1 = 2A - 5$$
$$2A = 6$$
$$A = 3$$

$\therefore$ $\underline{\underline{A = 3,\ B = 4 \text{ and } C = -1.}}$

Since it is an identity in x, the equality holds for any value of x.

Substitute the value of C into the equation.

Select $x = 1$ to make $A(x-1)(x-2)$ $+B(x-1) = 0$ so that the value of C can be found.

Select $x = 2$ to make $A(x-1)(x-2) = 0$ so that the value of B can be found.

Substitute the values of B and C.

3.2 Division of Polynomials

We shall show two methods to perform the division of polynomials. They are by long division and synthetic division.

Example 3

Find the quotient and the remainder when $6x^3-7x^2+3x-5$ is divided by $2x+1$.

Solution

Method 1 (Long Division)

$$2x+1\overline{\smash{\big)}\,6x^3-7x^2+3x-5}$$

In this example, $6x^3-7x^2+3x-5$ is called the dividend and $2x+1$ is called the divisor.

$$\begin{array}{rl} & \;3x^2 \\ 2x+1 \big) & \overline{6x^3-7x^2+3x-5} \\ - & \underline{6x^3+3x^2} \end{array}$$

First, arrange the dividend and divisor in descending powers of x.

Divide $6x^3$ by $2x$ and write down $3x^2$ on the top.

$$\begin{array}{rl} & \;3x^2-5x \\ 2x+1 \big) & \overline{6x^3-7x^2+3x-5} \\ - & \underline{6x^3+3x^2} \\ & -10x^2+3x \\ & -\underline{10x^2-5x} \\ & 8x-5 \end{array}$$

Multiply $2x+1$ by $3x^2$ and write the result (i.e. $6x^3+3x^2$) underneath the dividend.

$$\begin{array}{rl} & \;3x^2-5x+4 \\ 2x+1 \big) & \overline{6x^3-7x^2+3x-5} \\ - & \underline{6x^3+3x^2} \\ & -10x^2+3x \\ & -\underline{10x^2-5x} \\ & 8x-5 \\ & -\underline{8x+4} \\ & -9 \end{array}$$

Substract the result from the dividend, i.e. $-7x^2-3x^2=-10x^2$ and then bring down the next term, i.e. $3x$.

Continue the process until the degree of the remainder is less than the degree of the divisor.

Here we can see that the degree of the remainder is less than the degree of the divisor.

Hence the quotient is $\underline{\underline{3x^2-5x+4}}$ and the remainder is $\underline{\underline{-9}}$.

Method 2 (Synthetic Division)

Example 4

Find the quotient and the remainder when $3x^3 - x^2 + 2x - 5$ is divided by $x - 2$.

Solution

First, write ONLY the coefficients inside an inverted division symbol:
Make sure you leave room inside, underneath the row of coefficients, to write another row of numbers later.

$$\begin{array}{r|rrrr} & 3 & -1 & 2 & -5 \\ & & & & \\ \hline & & & & \end{array}$$

Let $x - 2 = 0 \Rightarrow x = 2$.
Put $x = 2$, at the left:
Take the first number inside, representing the leading coefficient, and carry it down, unchanged, to below the division symbol:

$$\begin{array}{r|rrrr} 2 & 3 & -1 & 2 & -5 \\ & & & & \\ \hline & 3 & & & \end{array}$$

Multiply this carry-down value by 2 and carry the result up into the next column:

$$\begin{array}{r|rrrr} 2 & 3 & -1 & 2 & -5 \\ & & 6 & & \\ \hline & 3 & & & \end{array}$$

Add down the column:
(i.e. $-1 + 6 = 5$)

$$\begin{array}{r|rrrr} 2 & 3 & -1 & 2 & -5 \\ & & 6 & & \\ \hline & 3 & 5 & & \end{array}$$

Multiply 5 by 2 and carry the result up into the next column:

$$\begin{array}{r|rrrr} 2 & 3 & -1 & 2 & -5 \\ & & 6 & 10 & \\ \hline & 6 & 5 & & \end{array}$$

Add down the column:
(i.e. $2 + 10 = 12$)

$$\begin{array}{r|rrrr} 2 & 3 & -1 & 2 & -5 \\ & & 6 & 10 & \\ \hline & 3 & 5 & 12 & \end{array}$$

Multiply 12 by 2 and carry the new result up into the last column:

$$\begin{array}{r|rrrr} 2 & 3 & -1 & 2 & -5 \\ & & 6 & 10 & 24 \\ \hline & 3 & 5 & 12 & \end{array}$$

This last carried-down value is the remainder.

$\therefore$ the quotient is $\underline{\underline{3x^3 + 5x + 12}}$ and the remainder is $\underline{\underline{19}}$.

$$\begin{array}{r|rrrr} 2 & 3 & -1 & 2 & -5 \\ & & 6 & 10 & 24 \\ \hline & \underbrace{3 \quad 5 \quad 12}_{\text{quotient}} & & & \underbrace{19}_{\text{remainder}} \end{array}$$

3.3 Division Algorithm for Polynomials

When a polynomial $P(x)$ is divided by a non-constant Divisor $D(x)$, the **Division Algorithm for Polynomials** can be written as follows.

$$\text{Polynomial} = \text{Divisor} \times \text{Quotient} + \text{Remainder}$$
$$P(x) = D(x) \times Q(x) + R(x)$$

Note that the Degree of $R(x) <$ the Degree of $D(x)$
and the Degree of the $Q(x) =$ Degree of $P(x) -$ the Degree of $D(x)$.

Refer to Example 3. When $3x^3 - x^2 + 2x - 5$ is divided by $x - 2$, the quotient is $3x^2 + 5x + 12$ and the remainder is 19. Using **Division Algorithm for Polynomials**, it can be written as

$$3x^3 - x^2 + 2x - 5 = (x-2)(3x^2 + 5x + 12) + 19$$

Example 5

Given that when the expression $6x^3 - 23x^2 + 16x + 10$ is divided by $(x-3)(3x-4)$, the remainder is $ax + b$. Find the values of a and b. Hence, state the remainder.

Solution

By the division of algorithm of polynomials

$$\text{Dividend} = (\text{Divisor})(\text{Quotient}) + \text{Remainder}$$

$$6x^3 - 23x^2 + 16x + 10 = (x-3)(3x-4)Q(x) + ax + b \text{ .... (1)}$$

Q(x) is denoted as Quotient.

Substitute $x = 3$ into (1):

$$6(3)^3 - 23(3)^2 + 16(3) + 10 = 3a + b$$
$$13 = 3a + b$$
$$b = 13 - 3a \text{ ....(2)}$$

Remark
Select a suitable value of x so that the product of (Divisor)(Quotient) is zero.
In this case, $x = 3$ and $\frac{4}{3}$ are used.

Express b in terms a.

Substitute $x = \frac{4}{3}$ into (1):

$$6\left(\frac{4}{3}\right)^3 - 23\left(\frac{4}{3}\right)^2 + 16\left(\frac{4}{3}\right) + 10 = \frac{4}{3}a + b$$
$$12a + 9b = 42$$
$$4a + 3b = 14 \text{ ....(3)}$$

Factor by 3.

Substitute (2) into (3):

$$\begin{aligned} 4a+3(13-3a) &= 14 \\ 4a+39-9a &= 14 \\ -5a &= -25 \\ a &= \underline{\underline{5}} \end{aligned}$$

Substitute $a=5$ into (2):

$$\begin{aligned} b &= 13-3(5) \\ b &= \underline{\underline{-2}} \end{aligned}$$

$\therefore$ the remainder is $\underline{\underline{5x-2}}$.

Subsituting the values of a and b into $ax+b$ to obtain the remainder.

3.4 Remainder Theorem

The Remainder Theorem states that if a polynomial $f(x)$ is divided by $(ax-b)$, where $a \neq 0$, then the remainder is $f\left(\frac{b}{a}\right)$.

Note that this theorem allows us to find the remainder when $f(x)$ is divided by $(ax-b)$ without performing the division. However, if we want to find the quotient, we still need to use the method of long division or synthetic division.

Example 6

Find the remainder when $8x^4-2x^3+5x-2$ is divided by

(a) $x-1$, **(b)** $x+2$, **(c)** $2x-1$.

Solution

Let $f(x)=8x^4-2x^3+5x-2$.

First denote the expression as $f(x)$.

Refer **(a)**.
To find the remainder when the $f(x)$ is divided by $x-1$ means we find the value of $f(1)$.

(a) $f(1)=8(1)^4-2(1)^3+5(1)-2$
$= \underline{\underline{9}}$

In **(a)**, set $x-1=0$ so that $x=1$.
Substitute $x=1$ into $f(x)$.

The value obtains is the remainder. (Here, the remainder is 9 when it is divided by $x-1$)

(b) $f(-2)=8(-2)^4-2(-2)^3+5(-2)-2$
$= \underline{\underline{132}}$

(c) $f\left(\frac{1}{2}\right)=8\left(\frac{1}{2}\right)^4-2\left(\frac{1}{2}\right)^3+5\left(\frac{1}{2}\right)-2$
$= \underline{\underline{\frac{3}{4}}}$

In **(b)**, set $x+2=0$, so that $x=-2$.
Substitute $x=-2$ into $f(x)$.

In **(c)**, set $2x-1=0$ so that $x=\frac{1}{2}$.

Substitute $x=\frac{1}{2}$ into $f(x)$.

3.5 Using Remainder Theorem to find unknown constants in a Polynomial

The Remainder Theorem can be used to solve unknown(s) in a polynomial.

Example 7

The expression $5x^3 + kx^2 + 5x - 9$ leaves a remainder of -7 when divided by $5x - 2$. Find the value of k.

Solution

Let $P(x) = 5x^3 + kx^2 + 5x - 9$.

Given $P\left(\frac{2}{5}\right) = -7$,

$$\therefore\ 5\left(\frac{2}{5}\right)^3 + k\left(\frac{2}{5}\right)^2 + 5\left(\frac{2}{5}\right) - 9 = -7$$

$$\frac{8}{25} + \frac{4}{25}k + 2 - 9 = -7$$

$$\frac{4}{25}k = -\frac{8}{25}$$

$$4k = -8$$

$$k = \underline{\underline{-2}}$$

Begin by denoting the expression as $P(x)$.

Refer to the question.
"... $5x^3 + kx^2 + 5x - 9$ leaves a remainder of -7 when divided by $5x - 2$."
It means that $P\left(\frac{2}{5}\right) = -7$.

From the divisor, set $5x - 2 = 0$ so that $x = \frac{2}{5}$.

Substitute $x = \frac{2}{5}$ into $P(x)$.

Solve for k.

Example 8

The expression $ax^3 + x^2 + bx$ leaves a remainder 3 when divided by $x + 1$ and a remainder 6 when divided by $x - 2$. Find the value of a and of b.

Solution

(a) Let $f(x) = ax^3 + x^2 + bx$

Given $f(-1) = 3$,

$$\therefore\ a(-1)^3 + (-1)^2 + b(-1) = 3$$

$$-a + 1 - b = 3$$

$$a + b = -2 \text{ .... (1)}$$

Begin by denoting the expression as $f(x)$.

Refer to the question.
"... $ax^3 + x^2 + bx$ has a remainder 3 when it is divided by $x + 1$..". It means that $f(-1) = 3$.

From the divisor, set $x + 1 = 0$ so that $x = -1$.

Substitute $x = -1$ into $f(x)$.

Also given $f(2) = 6$,

$\therefore a(2)^3 + (2)^2 + b(2) = 6$

$8a + 4 + 2b = 6$

$4a + b = 1$ ....(2)

$(2) - (1)$:

$3a = -3$

$a = \underline{\underline{-1}}$

Substitute $a = -1$ into (1):

$(-1) + b = -2$

$b = \underline{\underline{-3}}$

Refer to the question. "... expression is divided by $x - 2$, the remainder is 6 ...". It means that $f(2) = 6$.

From the divisor, set $x - 2 = 0$ so that $x = 2$.

Substitute $x = 2$ into $f(x)$.

Solve (1) and (2) simultaneously to find a and b.

3.6 Factor Theorem

The Factor Theorem states that if a polynomial $f(x)$ is divided by $(ax - b)$, where $a \neq 0$, the remainder is $f\left(\frac{b}{a}\right) = 0$. Then, $(ax - b)$ is a factor of the polynomial $f(x)$.

Example 9

Determine which of the following is a factor of $x^3 - 7x + 6$.

(a) $x - 1$ **(b)** $2x + 3$

Solution

(a) Let $f(x) = x^3 - 7x + 6$.

$f(1) = (1)^3 - 7(1) + 6$

$= 0$

$\therefore (x - 1)$ is a factor of $f(x)$.

(b) $f\left(-\frac{3}{2}\right) = \left(-\frac{3}{2}\right)^3 - 7\left(-\frac{3}{2}\right) + 6$

$= \frac{105}{8}$

Since $f\left(-\frac{3}{2}\right) \neq 0$, $\therefore (2x + 3)$ is not a factor of $f(x)$.

First denote the expression as $f(x)$.

In **(a)**, set $x - 1 = 0$ so that $x = 1$. Substitute $x = 1$ into $f(x)$.

Since the value obtains is zero, we say that $(x - 1)$ is a factor of $f(x)$.

In **(b)**, set $2x + 3 = 0$ so that $x = -\frac{3}{2}$. Substitute $x = -\frac{3}{2}$ into $f(x)$.

Since the value obtains is not zero, we say that $(2x + 3)$ is not a factor of $f(x)$.

3.7 Using Factor Theorem to find unknown constants in a Polynomial

Example 10

Find the value of k for which $2x-1$ is a factor of $16x^3-2x^2+kx-9$.

Solution

Let $f(x)=16x^3-2x^2+kx-9$.

Given $f\left(\frac{1}{2}\right)=0$,

$$\therefore\ 16\left(\frac{1}{2}\right)^3-2\left(\frac{1}{2}\right)^2+k\left(\frac{1}{2}\right)-9=0$$

$$2-\frac{1}{2}+\frac{k}{2}-9=0$$

$$7\frac{1}{2}=\frac{k}{2}$$

$\therefore\ k=15$

Refer to the question.
"... $2x-1$ is a factor of $16x^3-2x^2+kx-9$" means $16x^3-2x^2+kx-9$ leaves a remainder 0 when it is divided by $2x-1$.
In other words, $f\left(\frac{1}{2}\right)=0$.

Substitute $x=\frac{1}{2}$ into $f(x)$.

Solve for k.

Example 11

Given that the polynomial $x^4+x^3+ax^2-5x+b$ is divisible by x^2+x-12, find the value of a and of b.

Solution

Let $f(x)=x^4+x^3+ax^2-5x+b$

Gievn $x^2+x-12=(x-3)(x+4)$,

since $f(3)=0$,

$$\therefore\quad (3)^4+(3)^3+a(3)^2-5(3)+b=0$$

$$81+27+9a-15+b=0$$

$$9a+b=-93\text{........... (1)}$$

also $f(-4)=0$,

$$\therefore\quad (-4)^4+(-4)^3+a(-4)^2-5(-4)+b=0$$

$$256-64+16a+20+b=0$$

$$16a+b=-212\text{......... (2)}$$

Take $(1)-(2)$:

$$-7a=119$$

$$a=\underline{\underline{-17}}$$

Substitute $a=-17$ into (2):

$$16(-17)+b=-212$$

$$b=\underline{\underline{60}}$$

Analytical Thinking
The word "divisible" means the remainder is zero when f(x) is divided by $(x-a)$, i.e. $f(a)=0$.

Factorise the quadratic factors.

Replace x with 3.

Replace x with – 4.

Example 12

The expression $ax^3 + 3x^2 + bx - 2$, where a and b are constants, has a factor of $x - 1$ and leaves a remainder 20 when divided by $x - 2$. Find the values of a and b.

Solution

Let $P(x) = ax^3 + 3x^2 + bx - 2$

Given $P(2) = 20$,

$$\therefore \quad a(2)^3 + 3(2)^2 + b(2) - 2 = 0$$

$$8a + 12 + 2b - 2 = 0$$

$$4a + b = 5 \text{ .... (1)}$$

Also, $P(1) = 0$,

$$\therefore \quad a(1)^3 + 3(1)^2 + b(1) - 2 = 0$$

$$a + 3 + b - 2 = 0$$

$$a + b = -1 \text{ .... (2)}$$

$(2) - (1)$:

$$3a = 6$$

$$a = \underline{\underline{2}} \text{ .... (3)}$$

Substituting (3) into (1) to find b:

$$4(2) + b = 5$$

$$b = \underline{\underline{-3}}$$

Analytical Thinking:
To find the values of a and b, we need two pieces of information and express as two equations. Then solve the equations simultaneously.

Begin by denoting the expression as $P(x)$.

Given that $P(x) = ax^3 + 3x^2 + bx - 2$ leaves a remainder 20 when divided by $x - 2$. To write mathematically, it means $P(2) = 20$.

Substitute $x = 2$ into $P(x)$.

Also given that $P(x) = ax^3 + 3x^2 + bx - 2$ has a factor of $x - 1$, it means that $P(1) = 0$.

Substitute $x = 1$ into $P(x)$.

3.8 Factorising Cubic Polynomials

Consider the following

$$x^3 + 6x^2 + 11x + 6 = (x+1)(x+2)(x+3).$$

$(x+1)$, $(x+2)$ and $(x+3)$ are called the factors of $x^3 + 6x^2 + 11x + 6$.

The steps to factorise a cubic polynomial are as follows.

Step 1 : Write the polynomial in x as $f(x)$.
Step 2 : Use trial and error to find a factor of $f(x)$.
Step 3 : Use any of the following methods to find the remaining factors.
- **(a)** Long Division
- **(b)** Synthetic Division
- **(c)** Comparing the Coefficients

Example 13

Factorise $x^3 + 4x^2 + x - 6$ completely.

Solution

Let $f(x) = x^3 + 4x^2 + x - 6$.

By trial and error,

$$f(1) = (1)^3 + 4(1)^2 + (1) - 6 \\ = 0$$

$\therefore$ $(x-1)$ is a factor of $f(x)$.

Using long division

$$\begin{array}{r l} & x^2 + 5x + 6 \\ x-1 \,\big) & x^3 + 4x^2 + x - 6 \\ - & \underline{x^3 - x^2} \\ & 5x^2 + x \\ - & \underline{5x^2 - 5x} \\ & 6x - 6 \\ - & \underline{6x - 6} \\ & 0 \end{array}$$

$$\therefore\ x^3 + 4x^2 + x - 6 = (x-1)(x^2 + 5x + 6) \\ = \underline{\underline{(x-1)(x+2)(x+3)}}$$

Using synthetic division

$$\begin{array}{c|cccc} 1 & 1 & 4 & 1 & -6 \\ & \downarrow & 1 & 5 & 6 \\ \hline & 1 & 5 & 6 & 0 \end{array}$$

$$\therefore\ x^3 + 4x^2 + x - 6 = (x-1)(x^2 + 5x + 6) \\ = \underline{\underline{(x-1)(x+2)(x+3)}}$$

Step 1
Begin by writing the polynomial as $f(x)$.

Step 2
Use trial and error until you get the first factor.

Step 3
Use long division, synthetic division or comparing the coefficients to find the remaining factors.

From the long division, we obtain the quotient which is a quadratic factor.

Misconception
It is wrong to write as $x = 1$, $x = -2$ or $x = -3$ as we are not solving for x.

By comparing coefficients

Let the quotient be ax^2+bx+c.

$$x^3+4x^2+x-6=(x-1)(ax^2+bx+c)$$
$$=ax^3+bx^2-ax^2-bx-c$$

Compare the coefficients of x^3 :

$$a=1$$

Compare the constant terms:

$$c=6$$

Compare the coefficients of x^2 :

$$4=b-a$$
$$4=b-1$$
$$\therefore \quad b=5$$

So, $x^3+4x^2+x-6=(x-1)(x^2+5x+6)$

$$=\underline{\underline{(x-1)(x+2)(x+3)}}$$

Expand out the brackets on the RHS.

Remark
Instead of expanding out the bracket as shown, the values of a and c can written down directly as $(x)(ax^2)=ax^2$ so $a=1$ and $(-1)(c)=-6$ so $c=6$.
$\Rightarrow x^3+4x^2+x-6=(x-1)(x^2+bx+6)$

Replace a with 1 to find the value of b.

3.9 Solving Cubic Equations

The expression ax^3+bx^2+cx+d, where a, b, c and d are constants is called a cubic polynomial in x, and when $ax^3+bx^2+cx+d=0$ it is called a cubic equation.

To solve a cubic equation in x, follow these steps.

Given a cubic equation $ax^3+bx^2+cx+d=0$, where a,b,c and d are integers.

Step 1 : Write $\mathrm{f}(x)=ax^3+bx^2+cx+d$.

Step 2 : Use trial and error to find a factor of $\mathrm{f}(x)$.

Step 3 : Use long division, synthetic division or comparing coefficients to find the quotient $\mathrm{Q}(x)$ of $\mathrm{f}(x)$,

Step 4 : Solve for $\mathrm{Q}(x)$ to obtain the other values of x.

Example 14

Solve the cubic equation $x^3 - 7x + 6 = 0$.

Solution

Let $f(x) = x^3 - 7x + 6$

By trial and error,

$$\begin{aligned} f(2) &= (2)^3 - 7(2) + 6 \\ &= 0 \end{aligned}$$

$\therefore$ $(x-2)$ is a factor of $f(x)$.

By long division, we have

$$\begin{array}{r|l} & x^2 + 2x - 3 \\ \hline x-2 & x^3 + 0x^2 - 7x + 6 \\ & -\underline{(x^3 - 2x^2)} \\ & \quad 2x^2 - 7x \\ & \quad -\underline{(2x^2 - 4x)} \\ & \qquad -3x + 6 \\ & \qquad -\underline{(-3x + 6)} \\ & \qquad\qquad 0 \end{array}$$

$$\begin{aligned} \therefore \quad f(x) &= (x-2)(x^2 + 2x - 3) \\ &= (x-2)(x+3)(x-1) \end{aligned}$$

Given $x^3 - 7x + 6 = 0$

$(x-2)(x+3)(x-1) = 0$

$x = \underline{\underline{2}}, \ \underline{\underline{-3}}$ or $\underline{\underline{1}}$

Step 1
Write the polynomial as $f(x)$.

Remark
To obtain the first factor, we observe that the coefficient of x^3 is 1 and the constant term is 6. $\therefore$ the possible linear factors of $f(x)$ are $x \pm 2$, $x \pm 3$ and $x \pm 6$.

Step 2
Use trial and error to find the first factor.
Substitute $x = 2$ into $f(x)$.
(Here, we see that $f(2) = 0$,
$\therefore$ $x - 2$ is a factor $f(x)$.

Step 3
Use long division, synthetic division or comparing the coefficients to find the quotient $Q(x)$ of $f(x)$.
(Here, we use long division)

We include the missing term, $0x^2$, in the dividend so that the sum is laid correctly.

You can check your division here. $(x+2)$ is a factor of $f(x)$, so the remainder must be 0.

Factorise the quadratic factor.

Solve for x.

3.10 Sum and Difference of Cubes

We can factorise the Sum and Difference of Cubes using the following results.

$$(a^3+b^3)=(a+b)(a^2-ab+b^2)$$
$$(a^3-b^3)=(a-b)(a^2+ab+b^2)$$

Example 15

Factorise each of the following.

(a) $64x^3+343y^3$ **(b)** $(2x-1)^3-8$

Solution

(a) $64x^3+343y^3$

$=(4x)^3+(7y)^3$

$=(4x+7y)[(4x)^2-4x(7y)+(7y)^2]$

$=\underline{\underline{(4x+7y)(16x^2-28xy+49y^2)}}$

Apply the formula
$(a^3+b^3)=(a+b)(a^2-ab+b^2)$.
Replace a with $4x$ and b with $7y$.

(b) $(2x-1)^3-8$

$=(2x-1)^3-(2)^3$

$=[(2x-1)-2][(2x-1)^2+(2x-1)(2)+(2)^2]$

$=(2x-3)(4x^2-4x+1+4x-2+4)$

$=\underline{\underline{(2x-3)(4x^2+3)}}$

Apply the formula
$(a^3-b^3)=(a-b)(a^2+ab+b^2)$.
Replace a with $(2x-1)$ and b with 2.

Exercise 3

A Equality of Polynomials

1. If $(x+1)(x+3)(x+5)=Ax^3+Bx^2+Cx+D$, find the values of A, B, C and D.

2. Given that for all real values of x, $Ax^2+B(x-1)+C(x+2)^2=-4x+12$, find the values of A, B and C.

3. Given that $A(x+1)^2+B(x+1)(x-1)+C(x-1)^2=2x^2-4x+6$ is an identity in x, find the values of A, B and C.

4. Given that $3x^3+10x^2+22x+27$
$$=(x+2)(ax^2-x+4)+b(x+2)^2+c$$
for all values of x, find the values of a, b and c.

5. Given that
$$3x^3+2x^2+x-4=A(x-1)+B(x-1)(x+1)$$
$$+Cx(x^2-1)+D, \text{ for all real}$$
values of x, show that $D=2$ and find the values of A, B and C.

B Division of Polynomials

6. Find the quotient and the remainder in each of the following division of polynomials.
   (a) $(2x^2+3x-4)$ is divided by $(x-2)$
   (b) $(3x^3+8x^2+4x+200)$ is divided by $(x+5)$
   (c) $(5+4x-3x^2)$ is divided by$(3-x)$
   (d) $(6x^3-25x^2+24x-38)$ is divided by $(2x-7)$
   (e) (x^3+3x^2-4x+9) is divided by (x^2-3x+2)
   (f) (x^3+x^2-7) is divided by $(x-3)$

7. Given that the expression $x^4-x^3+5x^2+x+3$ leaves a remainder of $ax+b$ when it is divided by x^2+4, find the value of a and of b.

C Division Algorithm for Polynomials

8. When the polynomial $2x^3+x^2-9x-9$ is divided by $(x-2)(x+3)$, the remainder is $ax+b$. Therefore, we have
   $2x^3+x^2-9x-9=(x-2)(x+3)\mathrm{Q}(x)+ax+b$,
   where $\mathrm{Q}(x)$ is the quotient.
   (a) State the degree of $\mathrm{Q}(x)$.
   (b) Find the values of a and b by substituting suitable values of x.

9. The polynomial $\mathrm{f}(x)$ leaves remainders of 4 and 1 when divided by $x+2$ and $x-1$ respectively. Find the remainder when $\mathrm{f}(x)$ is divided by x^2+x-2.

10. When px^3-2x^2+rx+7 is divided by $(x-2)(x+1)$ the remainder is $15x+9$. Find the value of p and of r.

11. The function f is defined as $\mathrm{f}(x)=x^4-2x^3+x^2-3$. Given that $\mathrm{f}(x)=(x^2-1)\mathrm{Q}(x)+Ax+B$ for all values of x, where $\mathrm{Q}(x)$ is a polynomial, find the remainder when $\mathrm{f}(x)$ is divided by x^2-1.

12. When $x^4-2x^3-7x^2+7x+a$ is divided by x^2+2x-1, the quotient is x^2+bx+2 and the remainder is $cx+7$. Find the values of the constants a, b and c.

D Remainder and Factor Theorem

13. Find the remainder when x^3+2x^2-3x+1 is divided by
    (a) $x+1$, **(b)** $2x-1$, **(c)** x.

14. If $\mathrm{f}(x)=2x^3-3x^2-3x+2$, determine whether each of the following is a factor of $\mathrm{f}(x)$ by using factor theorem.
    (a) $x+1$ **(b)** $x-1$ **(c)** $2x+3$

15. The cubic expression $4x^3+kx-5$ leaves a remainder of 9 when it is divided by $(x-2)$. Find the value of k.

16. The expression x^3+ax^2+bx-3 leaves a remainder of 1 when divided by $x-1$ and a remainder of -9 when divided by $x+1$. Find the values of a and b.

17. The polynomial $\mathrm{f}(x)=3x^4+px^3+qx-43$ gives a remainder of $76x+113$ when divided by x^2-x-6. Find the value of p and of q.

18. The expression $2x^3+ax^2-b$, where a and b are integers, leaves a remainder of $4b-1$ when divided by $x+2$ and leaves a remainder of a^2-9 when divided by $x-3$. Find the values of a and b.

19. Find the value of k for which $x^2+5kx+k^2+5$ is exactly divisible by $x+2$ but not divisible by $x+3$.

20. The polynomial $\mathrm{f}(x)=ax^3+bx^2-11x+6$ leaves a remainder of 12 when divided by $x+1$. Given also that $x+2$ is a factor of $\mathrm{f}(x)$, evaluate a and b.

21. The expression $x^3 + px^2 + qx + 6$, where p and q are constants, is exactly divisible by $x^2 - 2x - 3$. Find
(a) the value of p and of k,
(b) the remainder when the expression is divided by $(x + 2)$.

22. It is given that $f(x) = x^3 + px^2 - 2x + 4\sqrt{3}$ has a factor of $(x + \sqrt{2})$ and the other two roots of the equation $f(x) = 0$ are $2\sqrt{3}$ and k, where $k > 0$. Find, leaving your answer in the surd form,
(a) the value of p and of k,
(b) the remainder when $f(x)$ is divided by $(x + \sqrt{3})$.

23. The remainder of $x^4 + 3x^2 - 2x + 2$ divided by $x + a$ is the square of the remainder when $x^2 - 3$ is divided by $x + a$. Calculate the possible values of a.

24. The polynomial $g(x) = px^3 + qx^2 - 7x + q$ leaves a remainder of R when it is divided by $x + 1$ and the expression $g(x) - 4$ leaves a remainder of $2R$ when it is divided by $x - 2$.
(a) Show that $10p + q = 32$.
(b) Given further that $pq = 6$ and $p > q$, find the value of p and of q.

25. The expression $x^3 - x^2 - 2x - 3$ has the same remainder when divided by $x + a$ and $x - 2a$. Find the non-zero values of a and the corresponding values of the remainder.

26. Given that $f(x) = 3(x^{2n-1}) - 4(x^{n+1}) + 7x + 6$, where n is an odd positive integer, find
(a) the remainder when $f(x)$ is divided by $x + 1$,
(b) the value of n for which $x - 3$ is a factor of $f(x)$.

27. The cubic polynomial $f(x)$ is such that the roots of the equation $f(x) = 0$ are 1, -2 and k. It is given that the coefficient of x^3 is 2 and when $f(x)$ is divided by $x - 3$, the remainder is -20. Find
(a) the value of k,
(b) the remainder when $f(x)$ is divided by $x + 1$.

28. The cubic polynomial f is such that the roots of $f(x) = 0$ are $-3, -1$ and 4 and $f(x)$ leaves a remainder of 12 when it is divided by $x + 2$.
(a) Express $f(x)$ as a cubic polynomial in x with integers coefficients.
(b) Find the remainder when $f(x)$ is divided by $2x + 3$.

29. Given that the term containing the highest power of x in a polynomial $f(x)$ is x^4 and two of the roots of the equation are -3 and 1. Given that $(x^2 - 4x + 5)$ is a quadratic factor of $f(x)$, find
(a) an expression for $f(x)$,
(b) the number of real roots of the equation $f(x) = 0$, justifying your answer,
(c) the remainder when $f(x)$ is divided by $2x - 1$.

30. The graph of a cubic polynomial function $P(x)$ cuts the x-axis at -1 and 2, and passes through the point (3, 12).
(a) Find the two possible expressions for $P(x)$.
(b) Find the possible values of remainder when $P(x)$ is divided by $3x + 1$.

E Factorising Cubic Polynomials

31. Factorise each of the following cubic polynomials completely.
(a) $x^3 - 4x^2 - x + 4$
(b) $x^3 + 6x^2 + 12x + 8$
(c) $5x^3 - 9x^2 - 17x - 3$

32. The function f is defined by $f(x) = x^3 - 19x + k$, where k is a constant.
(a) Given that $x - 2$ is a factor of $f(x)$, find the value of k.
(b) Using the value of k found in **(a)**, factorise $f(x)$ completely.

F Solving Cubic Equations

33. Solve the following equations.
(a) $x^3 - 2x^2 - 5x + 6 = 0$
(b) $x^3 - 3x + 2 = 0$

34. Solve the following equations, giving your answers in the simplest surd form where necessary.
 (a) $2x^3 - 21 = 19x(x-2)$
 (b) $x^3 - 5x^2 - 5x + 1 = 0$

35. Find the x-coordinates of the points of intersection of the curve $y = 4x^3 - 6$ with the straight line $y = 13x$.

36. The polynomial expression $2x^3 + ax^2 + bx + 3$ has a factor of $(2x-1)$ but leaves a remainder of 12 when divided by $x+1$.
 (a) Find the values of a and b.
 (b) Factorise the expression completely.
 (c) Hence, solve the equation $54x^3 + 3 = 3x(8-9x)$.

37. It is given that $x^2 - x - 2$ is a quadratic factor of the polynomial expression $x^4 + ax^3 - x^2 - bx - 12$.
 (a) Find the value of a and of b.
 (b) Factorise the polynomial expression completely.
 (c) Hence, solve the equation $e^{3y} + ae^{2y} - e^{y} - 12e^{-y} - b = 0$, where y is a real number, in exact form.

38. When the function $f(x) = 2x^3 + ax^2 + bx + 6$ is divided by $x^2 + x - 2$, the remainder is $-8x + 4$.
 (a) Find the value of a and of b.
 (b) Show that $(x-2)$ is a factor of $f(x)$ and hence, factorise $f(x)$ completely.
 (c) By using a suitable substitution, or otherwise, solve the equation $6y^3 - 13y^2 + y + 2 = 0$.

39. The remainder when $2x^3 + 2x^2 - 13x + 12$ is divided by $x + a$ is 3 times the remainder when it is divided by $x - a$.
 (a) Show that $2a^3 + a^2 - 13a + 6 = 0$.
 (b) Solve this equation completely.

40. The expressions $3x^3 - 2x^2 - kx + 6$ and $3x^3 + x^2 + (8-k)x + 10$ leave the same remainder when they are divided by $x + m$, where m is an integer.
 (a) Find the value of m. Hence, calculate the value of k if the remainder is -12.
 (b) Using the value of k found in **(a)**, solve the equation $3x^3 - 2x^2 - kx + 6 = 0$, expressing non-integer roots in the form $\dfrac{c+\sqrt{d}}{2}$, where c and d are integers.

G Sum and Difference of Cubes

41. Factorise the following.
 (a) $8m^3 + 125$
 (b) $27m^3 + (x+1)^3$
 (c) $3x^3y^3 - 192z^3$
 (d) $(2x-y)^3 - 729$
 (e) $[4(x+1)]^3 + (2x-1)^3$
 (f) $[2(x+2y)]^3 - [3(x-y)]^3$

42. (a) Factorise $t^6 - 64$ completely.
 (b) Hence find the values of t for which $2t^6 - 128 = (t^2+4)^2 - 4t^2$.

43. The equation $x - 2x^2 + 3 = 0$ has roots α and β.
 (a) Find the value of $\alpha^3 + \beta^3$.
 (b) Hence, find the quadratic equation whose roots are α^3 and β^3.

44. Factorise $n^3 + (n+1)^3$ completely. Hence show that the sum of the two cubes of two consecutive positive integers can never be divisible by 2.

45. Factorise $(2n+2)^3 - (2n)^3$ completely. Hence show that the difference between the two cubes of two consecutive positive even numbers is always divisible by 8.

4 Partial Fractions

Algebraic Fractions

An algebraic fraction is defined as the quotient of two non-zero polynomials, i.e. $\mathrm{F}(x)=\dfrac{\mathrm{P}(x)}{\mathrm{Q}(x)}$, $\mathrm{Q}(x)\neq 0$.

For example, $\dfrac{1}{x+2}$ and $\dfrac{x-1}{x^2+3x+4}$.

Proper and Improper Algebraic Fractions

Consider P(x) and Q(x) are polynomials in x,

(a) when the highest degree of P(x) is less than the highest degree of Q(x),

i.e. (highest degree of P(x)) < (highest degree of Q(x)), we call $\dfrac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ a proper algebraic fraction.

For example, $\dfrac{x}{x^2+2}$ and $\dfrac{2x+1}{x^2+x+4}$ are proper algebraic fractions.

(b) when the highest degree of P(x) is more than or equal to the highest degree of Q(x),

i.e. (highest degree of P(x)) ≥ (highest degree of Q(x)), we call $\dfrac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ an improper algebraic fraction.

For example, $\dfrac{x}{x+2}$ and $\dfrac{2x^3+1}{x^2+x+4}$ are improper algebraic fractions.

4.1 Decomposing an Algebraic Fraction into Partial Fractions

The steps of decomposing an algebraic fraction into partial fractions are as follows.

Step 1 : Determine if the algebraic fraction is proper or improper. If it is improper, an additional step of performing long division

Step 2 : Factorise the polynomial in the denominator completely and if it is required, collect repeated factors as powers.

Step 3 : Write down the partial fraction(s) arising from each collected factor in the denominator according to the cases as shown on the next page.

Case	Types of Factors	Algebraic Fractions	Corresponding Partial Fractions
(A)	Linear factors	$\frac{1}{(ax+b)(cx+d)}$	$\frac{A}{(ax+b)}+\frac{B}{(cx+d)}$
(B)	Linear repeated factors	$\frac{1}{(ax+b)(cx+d)^2}$	$\frac{A}{(ax+b)}+\frac{B}{(cx+d)}+\frac{C}{(cx+d)^2}$
(C)	Linear and Quadratic factors	$\frac{1}{(ax+b)(cx^2+d)}$	$\frac{A}{(ax+b)}+\frac{Bx+C}{(cx^2+d)}$

Step 4 : Solve for the unknown constants (A, B, ...) by
- **(i)** substituting suitable values for x; or
- **(ii)** equating coefficients of the same powers of x; or
- **(iii)** a combination of **(i)** and **(ii)**.

Decomposing an Algebraic Fraction with Linear Factors in the Denominator into Partial Fractions

Example 1

Express $\frac{2-11x}{1-x-6x^2}$ in partial fractions.

Solution

$$\frac{2-11x}{1-x-6x^2}=\frac{2-11x}{(2x+1)(1-3x)}$$

Let $\frac{2-11x}{(2x+1)(1-3x)}=\frac{A}{2x+1}+\frac{B}{1-3x}$

$$2-11x=A(1-3x)+B(2x+1) \text{ ......(1)}$$

Substitute $x=-\frac{1}{2}$ into (1):

$$2-11\left(-\frac{1}{2}\right)=A\left[1-3\left(-\frac{1}{2}\right)\right]+0$$

$$\frac{15}{2}=\frac{5}{2}A$$

$$A=3$$

Substitute $x=\frac{1}{3}$ into (1):

$$2-11\left(\frac{1}{3}\right)=0+B\left[2\left(\frac{1}{3}\right)+1\right]$$

$$-\frac{5}{3}=\frac{5}{3}B$$

$$B=-1$$

$$\therefore\ \frac{2-11x}{(2x+1)(1-3x)}=\underline{\underline{\frac{3}{2x+1}-\frac{1}{1-3x}}}$$

Note that $\frac{2-11x}{1-x-6x^2}$ is proper but the polynomial in the denominator is not factorised.

On the LHS, factorise the quadratic expression in the denominator.

Next, check the factors on the RHS. (Here, it belongs to Case A)

Refer to the table above.

Multiply by $(2x+1)(1-3x)$ on each side.

Substitute suitable values for x which make one term on the RHS of (1) zero, leaving us with one unknown.

(Here, we choose $x=-\frac{1}{2}$ and $x=\frac{1}{3}$)

Remark

Alternatively, we may also equate the coefficients of x and constant terms to find A and B, but such is not the efficient method for our given case.

Decomposing an Algebraic Fraction with a non - factorisable Quadratic Factor, in the Denominator into Partial Fractions

Example 2

Express $\dfrac{7x^2-4x+5}{(x-2)(x^2+1)}$ in partial fractions.

Note that $\dfrac{7x^2-4x+5}{(x-2)(x^2+1)}$ is proper and the factors in the denominator is completely factorised.

Solution

Let $\dfrac{7x^2-4x+5}{(x-2)(x^2+1)}=\dfrac{A}{(x-2)}+\dfrac{Bx+C}{(x^2+1)}$

$7x^2-4x+5=A(x^2+1)+(Bx+C)(x-2)$ .... (1)

Substitute $x=2$ into (1):

$7(2)^2-4(2)+5=A(2^2+1)+0$

$25=5A$

$A=5$

Substitute $x=0$ into (1):

$7(0)^2-4(0)+5=A(0^2+1)+(0+C)(0-2)$

$5=A+C(-2)$

$5=5-2C$

$0=-2C$

$C=0$

Compare coefficients of x^2 of (1),

$7=A+B$

$7=5+B$

$B=2$

$\therefore\ \dfrac{7x^2-4x+5}{(x-2)(x^2+1)}=\underline{\underline{\dfrac{5}{x-2}+\dfrac{2x}{x^2+1}}}$

Observe the factors in the denominator on the LHS.
(Here, we see that it belongs to Case C)

Multiply by $(x-2)(x^2+1)$ on each side.

Common Mistake
$(Bx+C)(x-2)\neq Bx+C(x-2)$.

Step 4(i)
Substitute suitable values for x which makes one term on the RHS of (1) zero, leaving us with one unknown.

Substitute $A=5$ into the equation which leads to find C.

Substitute $A=5$ into the equation which leads to find B.

Decomposing an Algebraic Fraction with Repeated Factors in the Denominator into Partial Fractions

Example 3

Express $\dfrac{2}{(x-1)(x^2-1)}$ in partial fractions.

> Note that $\dfrac{2}{(x-1)(x^2-1)}$ is proper, but the factors in the denominator is not completely factorised.

> Common Mistake
> It is not correct to do the following.
> $\dfrac{2}{(x-1)(x^2-1)} \neq \dfrac{A}{x-1} + \dfrac{Bx+C}{(x^2-1)}$.
> This is because x^2-1 is not a non-factorisable quadratic factor.

Solution

$$\frac{2}{(x-1)(x^2-1)} = \frac{2}{(x-1)(x-1)(x+1)}$$

$$= \frac{2}{(x+1)(x-1)^2}$$

> Factorise expression in the denominator on the LHS completely.

Let $\dfrac{2}{(x+1)(x-1)^2} = \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$

$$2 = A(x-1)^2 + B(x+1)(x-1) + C(x+1) \text{.....(1)}$$

> Multiply by $(x+1)(x-1)^2$ on each side.

Substitute $x = 1$ into (1):

$$2 = A(1-1)^2 + B(1+1)(1-1) + C(1+1)$$
$$2 = 0 + 0 + C(1+1)$$
$$2 = 2C$$
$$\therefore \quad C = 1$$

> Observe (1). Think of a x-value to substitute such that one term on the RHS of (1) becomes zero, leaving with one unknown constant to solve. (Here, we choose $x = 1$ to find C and $x = -1$ to find A.

Substitute $x = -1$ into (1):

$$2 = A(-1-1)^2 + B(-1+1)(-1-1) + C(-1+1)$$
$$2 = A(-2)^2 + 0 + 0$$
$$2 = 4A$$
$$\therefore \quad A = \frac{1}{2}$$

Substituting $x = 0$ into (1):

$$2 = A(0-1)^2 + B(0+1)(0-1) + C(0+1)$$
$$2 = A(-1)^2 + B(1)(-1) + C(1)$$
$$2 = A - B + C$$
$$2 = \frac{1}{2} - B + 1$$
$$\therefore \quad B = -\frac{1}{2}$$

> Substitute the value of A and of C to find B.

$$\therefore \quad \frac{2}{(x-1)(x^2-1)} = \underline{\underline{\frac{1}{2(x+1)} - \frac{1}{2(x-1)} + \frac{1}{(x-1)^2}}}$$

Decomposing an Improper Algebraic Fraction into Partial Fractions

Example 4

Express $\dfrac{x^3-2x^2-25x+53}{25-x^2}$ in partial fractions.

> Note that $\dfrac{x^3-2x^2-25x+53}{25-x^2}$ is improper, so we need to perform long division.

Solution

$$
\begin{array}{r|l}
 & -x+2 \\
\hline
-x^2+25 & x^3-2x^2-25x+53 \\
 - & x^3 \qquad\quad -25x \\
\hline
 & -2x^2+0+53 \\
 - & -2x^2+0+50 \\
\hline
 & \qquad\qquad\qquad 3
\end{array}
$$

> Perform long division.

$$\frac{x^3-2x^2-25x+53}{25-x^2}=-x+2+\frac{3}{25-x^2}$$

> Express $\dfrac{3}{25-x^2}$ in partial fractions.

Let $\quad \dfrac{3}{25-x^2}=\dfrac{A}{5-x}+\dfrac{B}{5+x}$

> Multiply by $(5+x)(5-x)$ on each side.

$$3=A(5+x)+B(5-x) \text{ .... (1)}$$

> Select a x-value which results one term on the RHS of (1) zero, leaving with one unknown.
> (Here, we choose $x=5$ and $x=-5$.)

Substitute $x=5$ into (1):

$$3=A(5+5)+B(5-5)$$
$$3=10A+0$$
$$\therefore \quad A=\frac{3}{10}$$

> Substitute $x=5$ into (1) leading to find A.

Substitute $x=-5$ into (1):

$$3=A(5-5)+B(5+5)$$
$$3=0+10B$$
$$\therefore \quad B=\frac{3}{10}$$

> Substitute $x=-5$ into (1) leading to find B.

$$\therefore \quad \frac{x^3-2x^2-25x+53}{25-x^2}=\underline{\underline{-x+2+\frac{3}{10(5-x)}+\frac{3}{10(5+x)}}}$$

Exercise 4

A Partial Fractions

1. Express the following in partial fractions.

(a) $\dfrac{6x-2}{(x-3)(x+1)}$ **(b)** $\dfrac{7-3x}{x^2-3x-4}$

(c) $\dfrac{2x-1}{x(4x-3)^2}$ **(d)** $\dfrac{8+3x-2x^2}{(x-1)(x+2)^2}$

(e) $\dfrac{5x^2-2x+3}{x(2x^2+1)}$ **(f)** $\dfrac{x^2+3x}{(x-2)(x^2+3)}$

2. Express the following in partial fractions.

(a) $\dfrac{x^3-2x^2-25x+53}{25-x^2}$ **(b)** $\dfrac{6x^2-20x+10}{3x^2-2x}$

(c) $\dfrac{2x^3+2x-1}{2x^2-x-1}$

Chapter 4.indd-cs5.indd 47 2015-11-16 22:10:38

3. Express $\dfrac{x-5}{x^2-4x+4}$ in partial fractions.

4. The function f is such that $\mathrm{f}(x)=\dfrac{2x}{(x-1)(x^2-1)}$, and $\mathrm{f}(x)$ may be expressed in the form $\dfrac{A}{x+1}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$, where A, B and C are constants. Find the values of A, B and C.

5. Factorise x^3+3x^2-4 and hence express $\dfrac{2x^3+6x^2+1}{x^3+3x^2-4}$ as the sum of a polynomial and a proper fraction.

6. Find the values of A, B and C such that $3x^3+Ax^2+12x+8=(3x+B)(Cx^2+4)$. Hence, or otherwise express $\dfrac{8x^2-5x+2}{3x^3+Ax^2+12x+8}$ in partial fractions.

7. The quadratic function f(x) is defined as $\mathrm{f}(x)=ax^2+bx+c$, where a, b and c are non-zero real numbers.
   (a) It is given that f(x) leaves a remainder of -12 when divided by $x-1$ and of 10 when divided by $x+1$. Given further that, when $\mathrm{f}(x)=0$, the sum of the roots of this equation $\dfrac{11}{3}$, find the value of a, of b and of c.
   (b) If $\mathrm{g}(x)=6x^2-11x-26$, express $\dfrac{\mathrm{g}(x)}{\mathrm{f}(x)}$ in partial fractions.

8. It is given that $\mathrm{f}(x)=(2x-1)(x-3)(x+2)$.
   (a) Express $\mathrm{f}(x)$ in the form of Ax^3+Bx^2+Cx+D, giving the values of the constants A, B, C and D.
   (b) Find the value of the constant a, given that $(x+3)$ is a factor of $\mathrm{f}(x)+ax$.
   (c) Express $\dfrac{x-3}{\mathrm{f}(x)}$ in partial fractions.

9. **(a)** The polynomial $\mathrm{f}(x)$ is defined by $\mathrm{f}(x)=2x^3+3x^2-18x+8$.
   (i) Use the Factor Theorem to show that $(2x-1)$ is a factor of $\mathrm{f}(x)$.
   (ii) Write $\mathrm{f}(x)$ in the form $(2x-1)(x^2+px+q)$, where p and q are integers to be determined.
   (b) Express $\dfrac{4x^2+16x}{2x^3+3x^2-18x+8}$ in partial fractions.

10. Given that $\dfrac{2x^3-2x^2-24x-7}{(x+3)(x-4)}=Ax+B+\dfrac{C}{x+3}+\dfrac{D}{x-4}$, where A, B, C and D are constants, find the values of A, B, C and D.

11. Divide $8x^3-12x^2-18x+15$ by $8x^3-12x^2-18x+27$. Hence, express $\dfrac{8x^3-12x^2-18x+15}{8x^3-12x^2-18x+27}$ in partial fractions.

12. **(a)** Express $\dfrac{1}{n(n+1)}$ in partial fractions.
   (b) Hence, evaluate
   $$\frac{1}{1(2)}+\frac{1}{2(3)}+\frac{1}{3(4)}+\ldots+\frac{1}{98(99)}+\frac{1}{99(100)}.$$

13. Express each of the following in partial fractions.
   (a) $\dfrac{9-8x}{(2x-1)(3-x)}$ **(b)** $\dfrac{12}{(2x-1)(3-x)}$
   Hence find $\dfrac{(9x-8)^2}{(2x-1)^2(3-x)^2}$.