Paper 1 Solutions

1

$$\mathrm{f}(x)=\frac{ax+b}{x+c}$$

$\mathrm{f}(0)=4$

$\dfrac{b}{c}=4$

$b-4c=0 \qquad \ldots(1)$

$\mathrm{f}(2)=2$

$\dfrac{2a+b}{2+c}=2$

$2a+b-2c=4 \qquad \ldots(2)$

$\mathrm{f}(-3)=-0.5$

$\dfrac{-3a+b}{-3+c}=-0.5$

$-3a+b+0.5c=1.5 \qquad \ldots(3)$

From GC, $a=1, b=4, c=1$

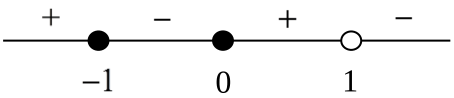
$$\mathrm{f}(x)=\frac{x+4}{x+1}=1+\frac{3}{x+1}$$

<u>Step 1</u>
Translation in the negative x-direction by 1 unit
<u>Step 2</u>
Scale parallel to the y-axis by a factor of 3
<u>Step 3</u>
Translation in the positive y-direction by 1 unit

2 (i)

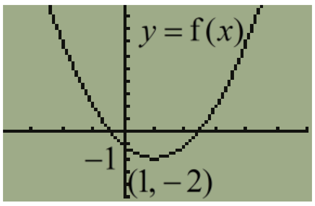
$\dfrac{x+x^2}{(1-x)^3}\geq 0$

$\dfrac{x(1+x)}{(1-x)^3}\geq 0$

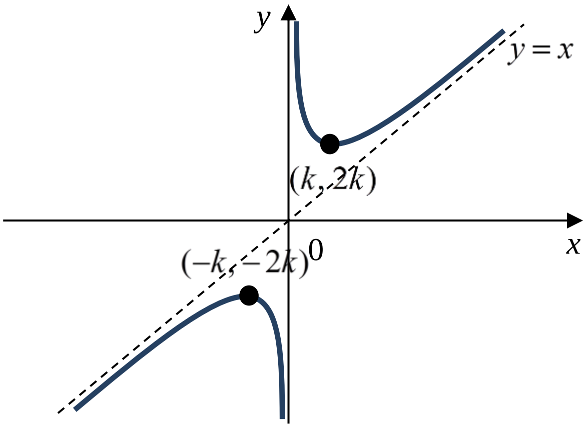


$\therefore x\leq -1$ or $0\leq x<1$

	(ii)	$\frac{x+x^2}{(1-x)^3}$ $=(x+x^2)(1-x)^{-3}$ $=(x+x^2)\left(1+(-3)(-x)+\frac{(-3)(-4)}{2!}(-x)^2+\ldots\right)$ $=(x+x^2)(1+3x+6x^2+\ldots)$ $=x+4x^2+9x^3+\ldots$
	(iii)	$\sum_{r=1}^{\infty}\frac{r^2}{2^r}=\frac{1^2}{2^1}+\frac{2^2}{2^2}+\frac{3^2}{2^3}+\ldots$ $=\frac{1}{2}+4\left(\frac{1}{2}\right)^2+9\left(\frac{1}{2}\right)^3+\ldots$ $=\frac{\frac{1}{2}+\left(\frac{1}{2}\right)^2}{\left(1-\frac{1}{2}\right)^3}$ $=6$
3	(i)	$y=\tan(e^x-1)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\sec^2(e^x-1)\times e^x$ $=(1+\tan^2(e^x-1))e^x$ $=(1+y^2)e^x$
	(ii)	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=(1+y^2)e^x+2y\frac{\mathrm{d}y}{\mathrm{d}x}e^x$ $=\frac{\mathrm{d}y}{\mathrm{d}x}+2y\frac{\mathrm{d}y}{\mathrm{d}x}e^x$ $=\frac{\mathrm{d}y}{\mathrm{d}x}(1+2ye^x)$
		When $x=0$, $y=0$ $\frac{\mathrm{d}y}{\mathrm{d}x}=1$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=1$ $\therefore \tan(e^x-1)=0+(1)x+(1)\frac{x^2}{2!}$ $=x+\frac{1}{2}x^2+\ldots$
		Required equation of tangent is $y=x$

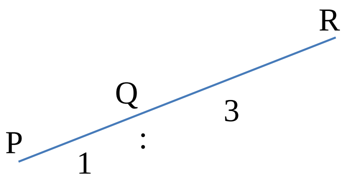
4		$\frac{z^3+8}{8-z^3}=\mathrm{i}$ $z^3+8=8\mathrm{i}-\mathrm{i}z^3$ $z^3=\frac{8(\mathrm{i}-1)}{1+\mathrm{i}}$ $z^3=8\mathrm{i}$ $\quad=8e^{\mathrm{i}\left(\frac{\pi}{2}+2n\pi\right)}$ $z=2e^{\mathrm{i}\left(\frac{\pi}{6}+\frac{2n\pi}{3}\right)}$, $n=0,\pm1$ $\quad=2e^{\frac{\pi}{6}\mathrm{i}},2e^{\frac{5\pi}{6}\mathrm{i}},2e^{-\frac{\pi}{2}\mathrm{i}}$
		$\frac{1+8w^6}{8w^6-1}=\mathrm{i}$ $\frac{\left(\frac{1}{w^6}\right)+8}{8-\left(\frac{1}{w^6}\right)}=\mathrm{i}$ $\therefore$ let $\frac{1}{w^2}=z$ Then $\frac{1}{w^2}=2e^{\frac{\pi}{6}\mathrm{i}},2e^{\frac{5\pi}{6}\mathrm{i}},2e^{-\frac{\pi}{2}\mathrm{i}}$ $w^2=\frac{1}{2}e^{-\frac{\pi}{6}\mathrm{i}},\frac{1}{2}e^{-\frac{5\pi}{6}\mathrm{i}},\frac{1}{2}e^{\frac{\pi}{2}\mathrm{i}}$ $w=\pm\frac{1}{\sqrt{2}}e^{-\frac{\pi}{12}\mathrm{i}},\pm\frac{1}{\sqrt{2}}e^{-\frac{5\pi}{12}\mathrm{i}},\pm\frac{1}{\sqrt{2}}e^{\frac{\pi}{4}\mathrm{i}}$
5	(i)	 Since the line $y=0$ cuts the curve $y=\mathrm{f}(x)$ more than once, f is not one–to–one and thus f has no inverse. Least $k=1$
	(ii)	Let $y=x^2-2x-1$ $(x-1)^2-2=y$ $x-1=\sqrt{y+2}$ $\qquad(\because x\geq1)$ $x=1+\sqrt{y+2}$ $\therefore \mathrm{f}^{-1}(x)=1+\sqrt{x+2},\quad x\geq-2$

	(iii)	$\mathrm{ff}^{-1}(x) = \mathrm{f}(\mathrm{f}^{-1}(x))$ $= \mathrm{f}(1+\sqrt{x+2})$ $= (1+\sqrt{x+2}-1)^2 - 2$ $= x+2-2$ $= x$
		$\mathrm{gf}(x) = x^2$ $\mathrm{gf}(\mathrm{f}^{-1}(x)) = (\mathrm{f}^{-1}(x))^2$ $\mathrm{g}(x) = (1+\sqrt{x+2})^2$ $= 1+2\sqrt{x+2}+x+2$ $= 3+2\sqrt{x+2}+x$
6	(i)	$\mathrm{h}(-x) = \dfrac{(-x)^2+k^2}{-x}$ $= -\dfrac{x^2+k^2}{x}$ $= -\mathrm{h}(x)$ Thus h is an odd function
		$\mathrm{gh}(-x) = \mathrm{g}(\mathrm{h}(-x))$ $= \mathrm{g}(-\mathrm{h}(x))$ ($\because$ h is odd) $= -\mathrm{gh}(x)$ ($\because$ g is odd) Thus, gh is an odd function.
	(ii)	$\dfrac{x^2+k^2}{x} = y$ $x^2+k^2 = xy$ $x^2 - yx + k^2 = 0$ $b^2 - 4ac \geq 0$ $(-y)^2 - 4(1)(k^2) \geq 0$ $y^2 \geq 4k^2$ $y \leq -2k$ or $y \geq 2k$ Required set = $\{y \in \mathbb{R} : y \leq -2k \text{ or } y \geq 2k\}$

	(iii)	
		$m > 0, m \neq 1$
7	(i)	$\frac{dP}{dt} = kP - m$, $(k, m > 0)$ $\int \frac{1}{kP-m}\,dP = \int 1\,dt$ $\frac{1}{k}\ln\lvert kP-m\rvert = t + c$ $\ln\lvert kP-m\rvert = kt + kc$ $kP - m = \pm e^{kc}e^{kt}$ $kP - m = Ae^{kt}$ $\quad (A = \pm e^{kc})$ When $t = 0$, $P = P_0$ $A = kP_0 - m$ $kP - m = (kP_0 - m)e^{kt}$ $kP = (kP_0 - m)e^{kt} + m$ $P = \left(P_0 - \frac{m}{k}\right)e^{kt} + \frac{m}{k}$
	(ii)	For decline in the population, $P_0 - \frac{m}{k} < 0$ i.e. $m > kP_0$
	(iii)	$P_0 = 8000000, k = 0.015, m = 200000$ $kP_0 = 0.015 \times 8000000 = 120000$ Since $m > kP_0$, the population was declining in that year.

8 (a)

By Ratio Theorem,



$$\frac{\mathbf{r}+3\mathbf{p}}{4}=\mathbf{q}$$

$$\mathbf{r}=4\mathbf{q}-3\mathbf{p}$$

Required length of projection

$$=\left|\overrightarrow{OR}\cdot\frac{\overrightarrow{OP}}{\left|\overrightarrow{OP}\right|}\right|$$

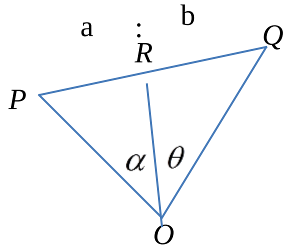
$$=\left|(4\mathbf{q}-3\mathbf{p})\cdot\frac{\mathbf{p}}{|\mathbf{p}|}\right|$$

$$=\left|4\mathbf{q}\cdot\frac{\mathbf{p}}{|\mathbf{p}|}-3\mathbf{p}\cdot\frac{\mathbf{p}}{|\mathbf{p}|}\right|$$

$$=\left|4\mathbf{q}\cdot\frac{\mathbf{p}}{|\mathbf{p}|}-\frac{3|\mathbf{p}|^2}{|\mathbf{p}|}\right|$$

$$=\left|\frac{4\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}|}-3|\mathbf{p}|\right|$$

(b)



By ratio theorem, $\mathbf{r}=\dfrac{a\mathbf{q}+b\mathbf{p}}{a+b}$

$$\frac{\mathbf{r}\cdot\mathbf{p}}{\mathbf{r}\cdot\mathbf{q}}=\frac{(a\mathbf{q}+b\mathbf{p})\cdot\mathbf{p}}{(a\mathbf{q}+b\mathbf{p})\cdot\mathbf{q}}$$

$$=\frac{a\mathbf{p}\cdot\mathbf{q}+b\mathbf{p}\cdot\mathbf{p}}{a\mathbf{q}\cdot\mathbf{q}+b\mathbf{p}\cdot\mathbf{q}}=\frac{a\mathbf{p}\cdot\mathbf{q}+b|\mathbf{p}|^2}{a|\mathbf{q}|^2+b\mathbf{p}\cdot\mathbf{q}}$$

When $\theta=\alpha$,

$$\frac{|\mathbf{r}||\mathbf{p}|\cos\theta}{|\mathbf{r}||\mathbf{q}|\cos\theta}=\frac{a|\mathbf{p}||\mathbf{q}|\cos2\theta+b|\mathbf{p}|^2}{a|\mathbf{q}|^2+b|\mathbf{p}||\mathbf{q}|\cos2\theta}$$

$$\frac{a|\mathbf{q}|\cos2\theta+b|\mathbf{p}|}{a|\mathbf{q}|+b|\mathbf{p}|\cos2\theta}=1$$

$$a|\mathbf{q}|\cos2\theta+b|\mathbf{p}|=a|\mathbf{q}|+b|\mathbf{p}|\cos2\theta$$

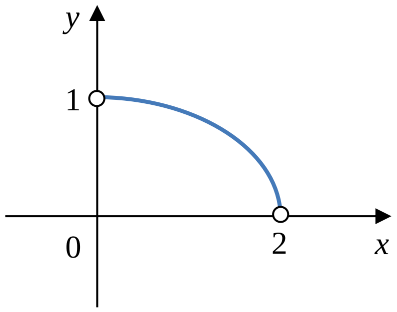
$$a|\mathbf{q}|(\cos2\theta-1)=b|\mathbf{p}|(\cos2\theta-1)$$

$$\frac{a}{b}=\frac{|\mathbf{p}|}{|\mathbf{q}|}$$

9	(i)	Let P_n be the statement $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$, $n \in \mathbb{Z}^+$ When $n=1$, $\text{LHS} = 1^3 = 1$ $\text{RHS} = \frac{1}{4}(1)^2(1+1)^2 = 1$ Since $\text{LHS} = \text{RHS}$, P_1 is true Assume that P_k is true for some $k \in \mathbb{Z}^+$ i.e. $\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$ To prove P_{k+1} is true i.e. $\sum_{r=1}^{k+1} r^3 = \frac{1}{4}(k+1)^2(k+2)^2$ $\sum_{r=1}^{k+1} r^3$ $= \sum_{r=1}^{k} r^3 + (k+1)^3$ $= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$ $= \frac{1}{4}(k+1)^2(k^2+4k+4)$ $= \frac{1}{4}(k+1)^2(k+2)^2$ Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, by Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$.
	(ii)	$u_r - u_{r-1} = r^3 - (r-1)^3$ $= r^3 - (r^3 - 3r^2 + 3r - 1)$ $= 3r^2 - 3r + 1$ $\sum_{r=1}^{n}(u_r - u_{r-1}) = \sum_{r=1}^{n}(3r^2 - 3r + 1)$ $\cancel{u_1} - u_0$ $+ \cancel{u_2} - \cancel{u_1}$ $\vdots$ $\qquad = 3\sum_{r=1}^{n} r^2 - 3\sum_{r=1}^{n} r + n$ $+ \cancel{u_{n-1}} - \cancel{u_{n-2}}$ $+ u_n - \cancel{u_{n-1}}$ $n^3 = 3\sum_{r=1}^{n} r^2 - 3 \times \frac{1}{2}n(n+1) + n$

		$3\sum_{r=1}^{n} r^2 = n^3+\frac{3}{2}n(n+1)-n$ $=\frac{1}{2}n(2n^2+3n+3-2)$ $=\frac{1}{2}n(n+1)(2n+1)$ $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$
	(iii)	$1^2\times 2+2^2\times 3+\ldots+50^2\times 51$ $=\sum_{r=1}^{50} r^2(r+1)$ $=\sum_{r=1}^{50}(r^3+r^2)$ $=\sum_{r=1}^{50} r^3+\sum_{r=1}^{50} r^2$ $=\frac{1}{4}(50^2)(51^2)+\frac{1}{6}(50)(51)(101)$ $=1668550$
10	(a) (i)	$(a+c)-(a+b)=c-b$ $(b+c)-(a+c)=b-a$ Since *a*, *b* and *c* are consecutive terms of AP, $c-b=b-a$ Hence $a+b, a+c$ and $b+c$ are in AP
	(a) (ii)	Since *a*, *b* and *c* are in AP $b-a=c-b$ $2b=a+c \qquad \ldots(1)$ Since *a*, *b* and *c* are in GP $\frac{b}{a}=\frac{c}{b}$ $b^2=ac \qquad \ldots(2)$ (1) gives $b=\frac{a+c}{2}$ Subs. $b=\frac{a+c}{2}$ into (2): $\left(\frac{a+c}{2}\right)^2=ac$

		$(a+c)^2 = 4ac$ $a^2 + 2ac + c^2 = 4ac$ $a^2 - 2ac + c^2 = 0$ $(a-c)^2 = 0$ $a - c = 0$ $a = c$ Subs. $a = c$ into (1): $2b = a + a$ $b = a$ Hence $a = b = c$
	(b) (i)	Required distance $= a + \frac{2}{3}a \times 2 + \left(\frac{2}{3}\right)^2 a \times 2 + \ldots + \left(\frac{2}{3}\right)^{n-1} a \times 2$ $= a + \frac{\frac{4}{3}a\left(1 - \left(\frac{2}{3}\right)^{n-1}\right)}{1 - \frac{2}{3}}$ $= a + 4a\left(1 - \left(\frac{2}{3}\right)^{n-1}\right)$ $= 5a - 4a\left(\frac{2}{3}\right)^{n-1}$
	(b) (ii)	Total distance travelled before the ball stops bouncing, L $= 5a$
		$5a - 4a\left(\frac{2}{3}\right)^{n-1} > 0.9 \times 5a$ $4\left(\frac{2}{3}\right)^{n-1} < 0.5$ $\left(\frac{2}{3}\right)^{n-1} < 0.125$ $n - 1 > \frac{\ln 0.125}{\ln \frac{2}{3}}$ $n > 6.13$ Hence the required least n is 7.
11	(i)	$\sin 2x = 2\sin x \cos x$ Differentiate both sides with respect to x

		$2\cos 2x = 2[\sin x(-\sin x) + \cos x(\cos x)]$ $\cos 2x = \cos^2 x - \sin^2 x$ $\cos 2x = 1 - \sin^2 x - \sin^2 x$ $\quad = 1 - 2\sin^2 x$
	(ii) (a)	
	(ii) (b)	Required area $= \int_0^2 y \, dx$ $= \int_{\frac{\pi}{2}}^{0} \sin\theta \cdot 2(-\sin\theta) \, d\theta$ $= -2\int_{\frac{\pi}{2}}^{0} \sin^2\theta \, d\theta$ $= 2\int_0^{\frac{\pi}{2}} \frac{1-\cos 2\theta}{2} \, d\theta$ $= \int_0^{\frac{\pi}{2}} (1-\cos 2\theta) \, d\theta$ $= \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$ $= \frac{\pi}{2}$
	(ii) (c)	$\frac{dx}{d\theta} = -2\sin\theta, \frac{dy}{d\theta} = \cos\theta$ $\frac{dy}{dx} = \cos\theta \times \frac{1}{-2\sin\theta} = -\frac{1}{2}\cot\theta$ Eq. of tangent at P: $y - \sin p = -\frac{1}{2}\cot p(x - 2\cos p)$ $y = -\frac{1}{2}x\cot p + \frac{\cos^2 p}{\sin p} + \sin p$
		Eq. of normal at P: $y - \sin p = 2\tan p(x - 2\cos p)$ $y = 2x\tan p - 3\sin p$ $OT \cdot ON$

		$=\left\|\dfrac{\cos^2 p}{\sin p}+\sin p\right\| \cdot \left\|-3\sin p\right\|$ $=\left\|-3(\cos^2 p+\sin^2 p)\right\|$ $=\left\|-3\right\|$ $=3$ (independent of p)
	(ii) (d)	Let Q be (x, y) $\dfrac{x+2\cos p}{2}=0$ $\dfrac{y+\sin p}{2}=-3\sin p$ $x=-2\cos p$ $y=-7\sin p$ $\cos p=-\dfrac{x}{2}$ $\sin p=-\dfrac{y}{7}$ $\sin^2 p+\cos^2 p=1$ $\dfrac{x^2}{4}+\dfrac{y^2}{49}=1, \quad x<0, y<0$ (eq. of locus of Q)