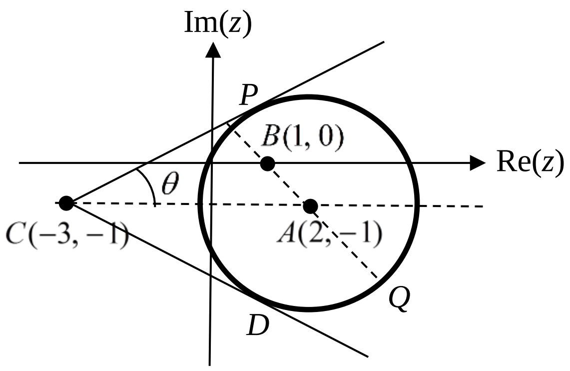
Paper 2 Solutions

1			 $\lvert z-(2-\mathrm{i})\rvert=\sqrt{5}$ $\lvert 1-z\rvert=\lvert z-1\rvert=$ distance of z from $(1, 0)$
			From diagram, $AB=\sqrt{2}$ $\therefore \sqrt{5}-\sqrt{2}\le\lvert z-1\rvert\le\sqrt{5}+\sqrt{2}$
			$\arg(\mathrm{i}z-1+3\mathrm{i})=\arg(\mathrm{i})+\arg(z-(-3-\mathrm{i}))$ From diagram, $\theta=\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$
			$\therefore \frac{\pi}{2}-\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)\le\arg(\mathrm{i}z-1+3\mathrm{i})\le\frac{\pi}{2}+\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$
2	(i)		$E=W\times AC+L\times CD$ $=W\sqrt{36+x^2}+L(15-x)$ $=W\sqrt{36+x^2}-Lx+15L$ $\frac{\mathrm{d}E}{\mathrm{d}x}=W\times\frac{1}{2}(36+x^2)^{-\frac{1}{2}}\times 2x-L$ $=\frac{Wx}{\sqrt{36+x^2}}-L$ At stationary value, $\frac{\mathrm{d}E}{\mathrm{d}x}=0$ and subs. $W=1.4L$ $\frac{1.4Lx}{\sqrt{36+x^2}}-L=0$ $1.4x=\sqrt{36+x^2}$ $1.96x^2=36+x^2$ $x=6.12$ (3s.f.)

	$\sqrt{37.5}^-$	$\sqrt{37.5}$	$\sqrt{37.5}^+$
$\dfrac{dE}{dx}$	–	0	+
Slope	\	__	/

So E is minimum when $x = 6.12$

(ii)

From (i), $\dfrac{dE}{dx} = \dfrac{Wx}{\sqrt{36+x^2}} - L = 0$

$$\frac{Wx}{\sqrt{36+x^2}} = L$$

$$\frac{W}{L} = \frac{\sqrt{36+x^2}}{x}$$

When $x = 4$, $\dfrac{W}{L} = \dfrac{\sqrt{52}}{4} = 1.80$ (3s.f.)

(iii)

$\dfrac{W}{L}$ is large implies that C is nearer to B

3 (a)

$$\int \frac{1}{4+9x^2}\,dx$$

$$= \int \frac{1}{2^2+(3x)^2}\,dx$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) + c$$

(b)

$$\int_0^2 \frac{x}{4+9x^2}\,dx$$

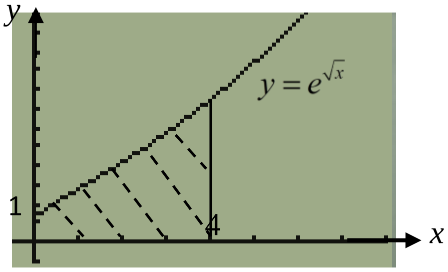
$$= \frac{1}{18}\int_0^2 \frac{18x}{4+9x^2}\,dx$$

$$= \frac{1}{18}\left[\ln(4+9x^2)\right]_0^2$$

$$= \frac{1}{18}(\ln 40 - \ln 4)$$

$$= \frac{1}{18}\ln 10$$

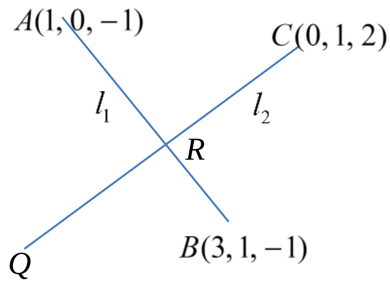
(c)

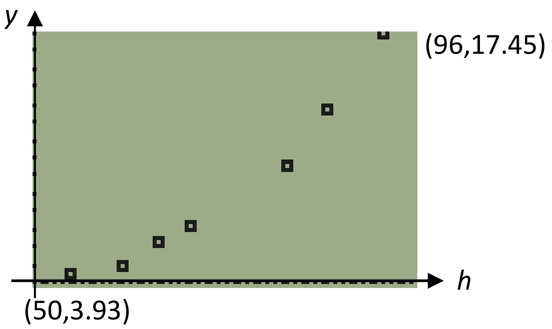


Volume

$$= \pi\int_0^4 y^2\ dx$$

$$= \pi\int_0^4 e^{2\sqrt{x}}\ dx$$

			$= \pi\int_0^2 e^{2t}\cdot 2t\ \mathrm{d}t$ $= 2\pi\int_0^2 te^{2t}\ \mathrm{d}t$ $= 2\pi\left[\left[\frac{1}{2}te^{2t}\right]_0^2 - \int_0^2 \frac{1}{2}e^{2t}\ \mathrm{d}t\right]$ $= 2\pi\left(e^4 - \left[\frac{1}{4}e^{2t}\right]_0^2\right)$ $= 2\pi\left(e^4 - \left(\frac{1}{4}e^4 - \frac{1}{4}\right)\right)$ $= 2\pi\left(\frac{3}{4}e^4 + \frac{1}{4}\right)$ $= \frac{1}{2}\pi(3e^4+1)$
4	(i)		 $\overrightarrow{AB} = \begin{pmatrix}2\\1\\0\end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix}-1\\1\\3\end{pmatrix}$ $\overrightarrow{AB}\times\overrightarrow{AC} = \begin{pmatrix}2\\1\\0\end{pmatrix}\times\begin{pmatrix}-1\\1\\3\end{pmatrix} = 3\begin{pmatrix}1\\-2\\1\end{pmatrix}$ Equation of plane *ABC* is $\mathbf{r}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix} = 1+0-1$ $\mathbf{r}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix} = 0$
	(ii)		$\overrightarrow{CR}\cdot\begin{pmatrix}2\\1\\0\end{pmatrix} = 0 \Rightarrow \begin{pmatrix}2\lambda+1\\\lambda-1\\-3\end{pmatrix}\cdot\begin{pmatrix}2\\1\\0\end{pmatrix} = 0$ $4\lambda+2+\lambda-1=0$ $\lambda = -\frac{1}{5} \qquad \therefore\ \overrightarrow{OR} = \begin{pmatrix}3/5\\-1/5\\-1\end{pmatrix}$

	(iii)		$BR \perp CQ$ Area $\Delta BCQ = (CR)(BR)$ $= \left(\left\|\overrightarrow{CR}\right\|\right)\left(\left\|\overrightarrow{BR}\right\|\right)$ $= \left\|\begin{pmatrix} \frac{3}{5} \\ -\frac{6}{5} \\ -3 \end{pmatrix}\right\| \left\|\begin{pmatrix} -\frac{12}{5} \\ -\frac{6}{5} \\ 0 \end{pmatrix}\right\|$ $= \sqrt{\frac{54}{5}}\left(\frac{6}{\sqrt{5}}\right)$ $= \frac{18\sqrt{6}}{5}$
	(iv)		Eqn of π_2 is $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 1$
	(i)		A random sample is a sample drawn from the population of 3000 students such that **<u>every student has an equal chance of being selected</u>**. The **<u>event that a student is chosen or not is independent of the event that any other student is being chosen or not</u>**. This may not provide a proportional representative sample because one of the groups may not be represented at all.
5	(ii)		A more appropriate sampling method is **stratified sampling method.** Divide student population into distinct strata i.e. Art, Science and Commerce streams. To obtain a sample of 120, r**andomly** select students from each stream with sample size proportional to the size of the stream.
	(i)		
6	(ii)		$r = 0.970$ From the scatter diagram, we observe that the data points seem to lie on a non-linear curve instead of a line. Hence, although r is close to 1, the best model for the relationship between y and h might not be linear.

	(iii)		$r_A = 0.760$ $r_B = 0.995$ Hence, the best model is $y = c + dh^3$ because the product moment correlation coefficient (0.995) is closer to 1.
	(iv)		From GC, the suitable regression line is $y = 0.00001772h^3 + 1.18266$ When h = 75cm , the weight of the baby = 8.66 kg The estimate is likely to be reliable as h = 75cm is within the data range. Moreover, the value of r is close to 1.
	(a)		Let $X \sim B(n, p)$. Note: $np = 4$, $npq = \frac{4}{3} \Rightarrow 4q = \frac{4}{3} \Rightarrow q = \frac{1}{3}$ $\Rightarrow p = \frac{2}{3}$ $\Rightarrow n = \frac{4}{p} = 6$ Hence, the largest value of X = 6
7			Since N is large, by CLT, $\overline{X} \sim N(4, \frac{4}{3N})$ approximately. Given that $P(\overline{X} \geq 3.8) > 0.9$ $P\left(Z \geq \frac{3.8-4}{\sqrt{\frac{4}{3N}}}\right) > 0.9$ $\frac{3.8-4}{\sqrt{\frac{4}{3N}}} < -1.281552$ $N > 54.7$ Therefore least N = 55
	(b)		Assumptions - Each cub has the same probability of reaching their matured age. - The success of each cub reaching his matured age is independent from the success of the other cubs.
			Let Y be the number of cubs which will reach their matured age out of 120 cubs. $Y \sim B(120, \frac{3}{4})$ Since n is large, $np = 90 > 5$, $nq = 30 > 5$, Therefore $Y \sim N(90, 22.5)$ approximately $P(Y \geq 90) \xrightarrow{c.c.} P(Y \geq 89.5)$ $= 0.542$

8	(i)		Let X be the volume of milk (ml) dispensed into a bottle. $X \sim \mathrm{N}(k, 8.5^2)$ $\mathrm{P}(X \geq 750) > 0.99$ $\mathrm{P}\left(Z \geq \frac{750-k}{8.5}\right) > 0.99$ $\Rightarrow \frac{750-k}{8.5} < -2.326$ $\Rightarrow k > 769.771$ The lowest mean volume that could be set is 770 ml.
8	(ii)		$\mathrm{H}_0 : \mu = 750$ $\mathrm{H}_1 : \mu < 750$ Under H_0 , $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0, 1)$ where $\sigma = 8.5$, $n = 100$ Significance level: 5% Reject H_0 if $Z < -1.6449$ $\Rightarrow \frac{\bar{x}-750}{8.5/\sqrt{100}} < -1.6449$ $\Rightarrow \bar{x} < 748.6$
	(iii)		$\mathrm{H}_0 : \mu = 750$ $\mathrm{H}_1 : \mu \neq 750$ Under H_0 , $T = \frac{\bar{Y}-\mu}{s/\sqrt{n}} \sim t(17)$ where $\bar{y} = \frac{\sum y}{18} = 754.5$ $s^2 = \frac{1}{18-1}\left(\sum y^2 - \frac{\left(\sum y\right)^2}{18}\right) = 49.147$ significance level: 1% From GC p-value = 0.0145 Since p-value = 0.0145 > 0.01, we do not reject H_0 and conclude that, at 1% significance level, there is no significant evidence that the manufacturer's claim is not justified.
	(i)		1) The typing errors on a randomly chosen page of a novel occur independently (or randomly or singly) 2) The average number of typing errors on a page is constant.
9	(ii)		Let X be the number of errors on a page of a novel. $X \sim \mathrm{Po}(0.4)$ Required probability = ${}^3C_1\, \mathrm{P}(X = 0)^2\, [1 - \mathrm{P}(X = 0)]$ $= 3(0.67032004)^2(1 - 0.67032004)$ $= 0.444$

	(iii)		Let W be the number of errors on n pages in a novel. $W \sim \text{Po}(0.4n)$ $\text{P}(W = 1) \geq 0.1$ $e^{-0.4n}(0.4n)^3 \quad 0.1$ From GC, $0.280 \leq n \leq 8.94$ Therefore the largest n is 8
	(iv)		Let Y ~Po(λ) P(Y = 2) = 2P(Y = 3) $e^{-\lambda}\frac{\lambda^2}{2!} = 2\left[e^{-\lambda}\frac{\lambda^3}{3!}\right]$ $\lambda = \frac{3}{2}$. Therefore E(Y) = $\frac{3}{2}$
	(v)		Let U and V be the number of errors on the first 100 pages of a novel and a Math textbook respectively. $U \sim \text{Po}(40)$, $V \sim \text{Po}(150)$ Since both $\lambda > 10$, $U \sim \text{N}(40, 40)$ approximately $V \sim \text{N}(150, 150)$ approximately $4U - V \sim \text{N}(10, 790)$ approximately $\text{P}(V \geq 4U) = \text{P}(4U - V \leq 0)$ $\longrightarrow \text{P}(4U - V < 0.5)$ $= 0.368$
	(a)	(i)	$\text{P}(B\|A') = 0.4 \Rightarrow \text{P}(B \cap A') = \text{P}(A')\text{P}(B\|A')$ $= (1 - 0.7)(0.4)$ $= 0.12$ $\text{P}(B) = \text{P}(B \cap A) + \text{P}(B \cap A') = 0.28 + 0.12 = 0.4$ Since $\text{P}(B\|A') = \text{P}(B)$, A and B are independent.
10		(ii)	$\text{P}\left[(A \cap B) \mid (A \cup B)\right] = \frac{\text{P}(A \cap B)}{\text{P}(A \cup B)}$ $= \frac{0.28}{0.7 + 0.12}$ $= \frac{14}{41}$ or 0.341
	(b)	(i)	Required Probability = 1 − P(Committee w/o couple) $= 1 - \frac{{}^{8}\text{C}_6}{{}^{10}\text{C}_6} = \frac{13}{15}$ <u>Alternative</u> Required Probability $= \frac{{}^{8}\text{C}_5\,{}^{2}\text{C}_1}{{}^{10}\text{C}_6} + \frac{{}^{8}\text{C}_4\,{}^{2}\text{C}_2}{{}^{10}\text{C}_6} = \frac{13}{15}$

		(ii)	P(more men than women | at least 2 men) $= \dfrac{P(4\,\text{men}, 2\,\text{women}) + P(5\,\text{men}, 1\,\text{woman})}{1 - P(1\,\text{man}, 5\,\text{women})}$ $= \dfrac{\dfrac{{}^5C_4\,{}^5C_2}{{}^{10}C_6} + \dfrac{{}^5C_5\,{}^5C_1}{{}^{10}C_6}}{\dfrac{{}^5C_5\,{}^5C_1}{{}^{10}C_6}}$ $= \dfrac{11}{41}$
		(iii)	P(Men and women sit alternatively) = $\dfrac{(3! \times 2 \times 3!)/2}{6!/2} = \dfrac{1}{10}$
		(iv)	P(Couple sit directly opposite to each other) $= \dfrac{(3 \times 2! \times 4!)/2}{6!/2}$ $= \dfrac{1}{5}$