Victoria Junior College
Preliminary Examinations 2014
H2 Mathematics H2 (9740) Paper 2

Solutions	
Q1(a) $\frac{1}{3}\pi r^2 h = k \Rightarrow r^2 = \frac{3k}{\pi h}$ $S^2 = \pi^2 r^2 \left(r^2 + h^2\right)$ $S^2 = \pi^2 \frac{3k}{\pi h}\left(\frac{3k}{\pi h} + h^2\right)$ $S^2 = \frac{9k^2}{h^2} + 3\pi kh$ Differentiating w.r.t. h, $2S\frac{\mathrm{d}S}{\mathrm{d}h} = -\frac{18k^2}{h^3} + 3\pi k$ For minimum S, $\frac{\mathrm{d}S}{\mathrm{d}h} = 0 \Rightarrow \frac{18k^2}{h^3} = 3\pi k$ $\Rightarrow h^3 = \frac{6k}{\pi}$ $\Rightarrow h = \sqrt[3]{\frac{6k}{\pi}}$ Hence $h = \sqrt[3]{\frac{6k}{\pi}}$ when S is minimum.	
Q1(b) Let h denote the perpendicular distance from C to AB and θ denote angle ACB. $\tan\frac{\theta}{2} = \frac{5}{h} \Rightarrow h = 5\cot\frac{\theta}{2}$. $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}\theta} \times \frac{\mathrm{d}\theta}{\mathrm{d}t}$ $\frac{\mathrm{d}h}{\mathrm{d}t} = \left(-5\operatorname{cosec}^2\frac{\theta}{2}\right)\left(\frac{1}{2}\right)\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)$ When the triangle is equilateral, $\theta = \frac{\pi}{3}$. $2 = \left(-5\operatorname{cosec}^2\frac{\pi}{6}\right)\left(\frac{1}{2}\right)\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)$ $2 = (-10)\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) \Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{1}{5}$ Hence the rate of change of angle ACB is $-\frac{1}{5}$ rad s $^{-1}$.	

Q2 (a)	

We consider a plane Π first, $\begin{pmatrix}1\\1\\2\end{pmatrix}$ is a normal to Π.

Equation of Π is $\mathbf{r}.\begin{pmatrix}1\\1\\2\end{pmatrix} = p$ and $B(p,0,0)$ lies in Π.

Perpendicular distance = $= 3\sqrt{6}$

$AP = \left|\overrightarrow{AB}.\hat{\underset{\sim}{n}}\right| = 3\sqrt{6}$

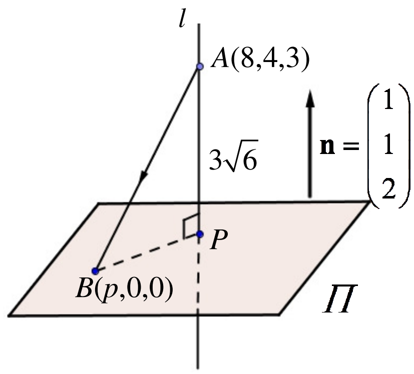
$\left|\begin{pmatrix}p-8\\-4\\-3\end{pmatrix}.\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\2\end{pmatrix}\right| = 3\sqrt{6}$

$\frac{1}{\sqrt{6}}\left|p-18\right| = 3\sqrt{6}$

$\left|p-18\right| = 18$

$p-18 = -18$ or 18

$p = 0$ or 36

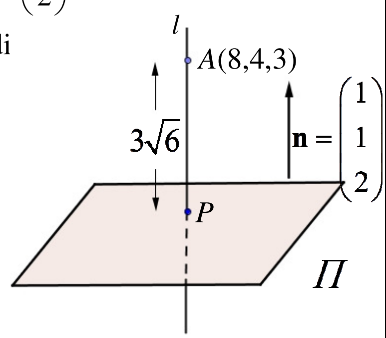


Equations of the planes are $x + y + 2z = 0$ and $x + y + 2z = 36$.

<u>Alternative Solution</u>

We consider a plane Π first, $\begin{pmatrix}1\\1\\2\end{pmatrix}$ is a normal to Π.

Let P be the foot of perpendi



$\overrightarrow{AP} = \pm 3\sqrt{6}\left[\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\2\end{pmatrix}\right] = \pm\begin{pmatrix}3\\3\\6\end{pmatrix}$

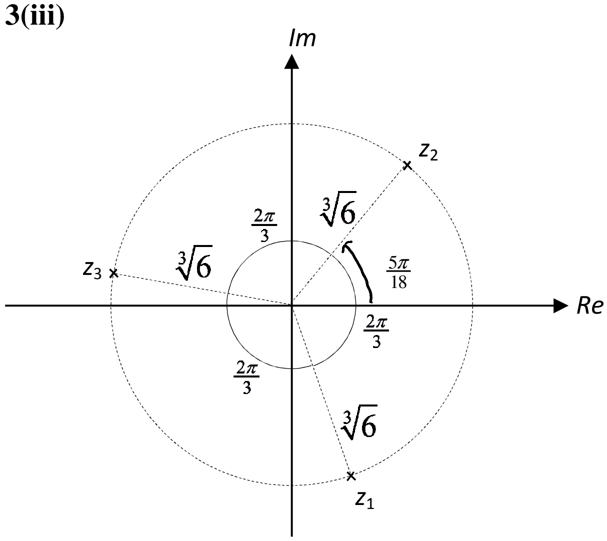
$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$= \begin{pmatrix}8\\4\\3\end{pmatrix} \pm \begin{pmatrix}3\\3\\6\end{pmatrix} = \begin{pmatrix}5\\1\\-3\end{pmatrix}$ or $\begin{pmatrix}11\\7\\9\end{pmatrix}$

$\mathbf{r}.\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}5\\1\\-3\end{pmatrix}.\begin{pmatrix}1\\1\\2\end{pmatrix} = 0$ or $\mathbf{r}.\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}11\\7\\9\end{pmatrix}.\begin{pmatrix}1\\1\\2\end{pmatrix} = 36$

Equations of the planes are $x + y + 2z = 0$ and $x + y + 2z = 36$.

2(b) (i) $l_1 : \mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}, \quad l_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 8 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ a \end{pmatrix}, \mu \in \mathbb{R}$ Consider $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ a \end{pmatrix}$, $\left.\begin{array}{r} 2+\lambda = 3+\mu \text{ ------- (1)} \\ -\lambda = 8+2\mu \text{ ------- (2)} \end{array}\right\}$ Solving (1) & (2), $\lambda = -2, \mu = -3$ $1+2\lambda = a\mu$ ------- (3) For skew lines, $\lambda = -2, \mu = -3$ do not satisfy (3). $-3 \neq -3a$, ie $a \neq 1$. $\therefore a \in \mathbb{R}, a \neq 1$	
2 (b)(ii) Since p and l_2 have no common point, l_2 is parallel to p. $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ are parallel to p. $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}$ is a normal to p. $\mathbf{r}.\begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} = 1$ $\therefore$ the required equation is $\mathbf{r}.\begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} = 1$	

3(i) $\left\|-3\sqrt{3}+3\mathrm{i}\right\| = \sqrt{27+9} = 6$ $\arg(-3\sqrt{3}+3\mathrm{i}) = \pi - \tan^{-1}\dfrac{3}{3\sqrt{3}} = \dfrac{5\pi}{6}$ $z^3 = 6e^{\mathrm{i}(\frac{5\pi}{6}+2k\pi)}$ $z = 6^{\frac{1}{3}}e^{\mathrm{i}(\frac{5\pi+12k\pi}{18})},\ k=0,\pm1$ $z_1 = 6^{\frac{1}{3}}e^{\mathrm{i}(-\frac{7\pi}{18})},\ z_2 = 6^{\frac{1}{3}}e^{\mathrm{i}(\frac{5\pi}{18})},\ z_3 = 6^{\frac{1}{3}}e^{\mathrm{i}(\frac{17\pi}{18})}$	
3(ii) $w^3 = -3-3\sqrt{3}\ \mathrm{i}$ $\mathrm{i}w^3 = -3\mathrm{i}+3\sqrt{3}$ $-\mathrm{i}w^3 = 3\mathrm{i}-3\sqrt{3} \Rightarrow (\mathrm{i}w)^3 = z^3$ Replace z with $\mathrm{i}w$. $\mathrm{i}w = e^{\mathrm{i}(\frac{\pi}{2})}w = 6^{\frac{1}{3}}e^{\mathrm{i}(-\frac{7\pi}{18})}, 6^{\frac{1}{3}}e^{\mathrm{i}(\frac{5\pi}{18})}, 6^{\frac{1}{3}}e^{\mathrm{i}(\frac{17\pi}{18})}$ $w = 6^{\frac{1}{3}}e^{\mathrm{i}(-\frac{8\pi}{9})}, 6^{\frac{1}{3}}e^{\mathrm{i}(-\frac{2\pi}{9})}, 6^{\frac{1}{3}}e^{\mathrm{i}(\frac{4\pi}{9})}$	
3(iii)  	
3(iv) $\arg(bz_2) = \arg(b) + \arg(z_2) = 0$ $\arg(b) = -\dfrac{5\pi}{18}$	

4(i)

$$a\left(\frac{1}{n^2}-\frac{1}{\left(n+2\right)^2}\right)=a\left(\frac{n^2+4n+4-n^2}{n^2\left(n+2\right)^2}\right)=\frac{4a\left(n+1\right)}{n^2\left(n+2\right)^2}$$

$$\therefore \frac{4a\left(n+1\right)}{n^2\left(n+2\right)^2}=\frac{n+1}{n^2\left(n+2\right)^2}\Rightarrow a=\frac{1}{4}$$

4(ii)

$$\begin{aligned}
\sum_{n=2}^{N}\frac{(n+1)}{n^2(n+2)^2} &= \frac{1}{4}\sum_{n=2}^{N}\left(\frac{1}{n^2}-\frac{1}{(n+2)^2}\right)\\
&= \frac{1}{4}\Bigg(\frac{1}{2^2} - \frac{1}{4^2}\\
&\qquad + \frac{1}{3^2} - \frac{1}{5^2}\\
&\qquad + \frac{1}{4^2} - \frac{1}{6^2}\\
&\qquad + \cdots\\
&\qquad + \frac{1}{(N-2)^2} - \frac{1}{N^2}\\
&\qquad + \frac{1}{(N-1)^2} - \frac{1}{(N+1)^2}\\
&\qquad + \frac{1}{N^2} - \frac{1}{(N+2)^2}\Bigg)\\
&= \frac{1}{4}\left(\frac{1}{4}+\frac{1}{9}-\frac{1}{(N+1)^2}-\frac{1}{(N+2)^2}\right)\\
&= \frac{13}{144}-\frac{1}{4(N+1)^2}-\frac{1}{4(N+2)^2}
\end{aligned}$$

4(iii)

As $N\to\infty$, $\dfrac{1}{4(N+1)^2}\to 0$ and $\dfrac{1}{4(N+2)^2}\to 0.$

$$\therefore \frac{13}{144}-\frac{1}{4(N+1)^2}-\frac{1}{4(N+2)^2}\to\frac{13}{144}.$$

Hence the series is convergent and $\displaystyle\sum_{n=2}^{\infty}\frac{(n+1)}{n^2(n+2)^2}=\frac{13}{144}$.

4(iv)

$$\sum_{n=3}^{\infty}\left(\frac{1}{n!}+\frac{\left(n-1\right)}{n^2\left(n-2\right)^2}\right)=\sum_{n=3}^{\infty}\frac{1}{n!}+\sum_{n=3}^{\infty}\frac{\left(n-1\right)}{n^2\left(n-2\right)^2}$$

$$=\left(\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots\right)+\left(\frac{2}{3^21^2}+\frac{3}{4^22^2}+\frac{4}{5^23^2}+\ldots\right)$$

$$=\left(\mathrm{e}-\left(1+1+\frac{1}{2!}\right)\right)+\left(\frac{2}{3^21^2}+\left(\frac{13}{144}\right)\right)=\mathrm{e}-\frac{35}{16}$$

5 The surveyor is likely to avoid throwing at the edge of the map. Hence not all the 500 seats have an equal chance of being selected.

<u>Alternative Solution</u>

After a seat is selected, the surveyor is likely to avoid the nearby seats. Hence the selections are not independent of one another.

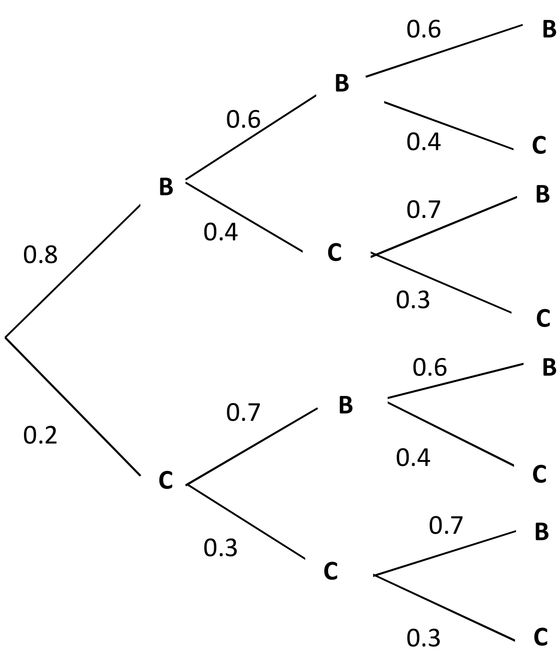
A more representative sample may be obtained by increasing the sample size.

A necessary assumption for the Poisson model to be valid is that the number of souvenirs sold in five minutes is independent of the number of souvenirs sold in another disjoint five-minute interval. This assumption may not be valid in this context since the spectators' decision to buy a souvenir could be influenced by the decision of people around them.

<u>Alternative Solution</u>

A necessary assumption for the Poisson model to be valid is that the average number of souvenirs sold in any five-minute interval is a constant. This assumption may not be valid in this context since more souvenirs may be sold during the intermission or before the game starts.

6 Let B and C denote the events in which a Beef and a Chicken Combo Meals are chosen respectively.



(i) P(BBB) = (0.8)(0.6)(0.6) = 0.288

6(ii) P(BCC) + P(CBC) +P(CCB)

= (0.8)(0.4)(0.3) + (0.2)(0.7)(0.4) + (0.2)(0.3)(0.7)

= 0.194

6(iii) P(Beef meal chosen in 2^{nd} visit|2 Chicken meals) $=\dfrac{\text{P(CBC)}}{\text{P}(\text{BCC})+\text{P}(\text{CBC})+\text{P}(\text{CCB})}$ $=\dfrac{(0.2)(0.7)(0.4)}{0.194}$ $=0.289$	
6(iv) No, as the probability of Mr Ong choosing a Beef Combo Meal on each visit is not a constant. **<u>Alternative Solution</u>** It is not possible to use a binomial distribution because the **events** that Mr Ong buys a beef combo meal on the 5 visits are not independent of one another.	
7 Let X ml and Y ml be the volume of a portion of bourbon whiskey and lemon juice respectively. $X \sim \text{N}(29.5, 0.6^2), \qquad Y \sim \text{N}(30.2, 0.4^2)$ $\text{E}\left(X_1+X_2+Y\right)=2(29.5)+(30.2)$ $=89.2$ $\text{Var}\left(X_1+X_2+Y\right)=2\left(0.6^2\right)+\left(0.4^2\right)$ $=0.88$ $\therefore X_1+X_2+Y \sim \text{N}(89.2, 0.88)$ $\text{P}\left(87<X_1+X_2+Y<95\right)$ $=0.990$ (3 s.f) The portions of bourbon whiskey and lemon juice are all independent of each other. Let \$$C$ be the cost price of a glass of bourbon whiskey. $C=0.05\left(X_1+X_2\right)+0.003Y$ $\text{E}\left(C\right)=\left(0.05\right)\left(2\right)(29.5)+\left(0.003\right)(30.2)$ $=3.0406$ $\text{Var}\left(C\right)=\left(0.05\right)^2\left(2\right)\left(0.6^2\right)+\left(0.003\right)^2\left(0.4^2\right)$ $=0.00180144$ $\therefore C \sim \text{N}(3.0406, 0.00180144)$ $\text{P}\left(\text{profit} \geq 7.50\right)=\text{P}\left(10.60-C \geq 7.50\right)$ $=\text{P}\left(C \leq 3.10\right)$ $=0.919$ (3 s.f)	

8(i) $X \sim \mathrm{B}(20,\ 0.15)$

$\mathrm{P}(X \geq 20 \times 0.15) = \mathrm{P}(X \geq 3)$

$= 1 - \mathrm{P}(X \leq 2)$

$= 0.595$ (3 s.f)

8ii) $\mathrm{P}(X > n) < 0.1$

$1 - \mathrm{P}(X \leq n) < 0.1$

$\mathrm{P}(X \leq n) > 0.9$

From GC,

n	$\mathrm{P}(X \leq n)$
4	$0.82985 < 0.9$
5	$0.93269 > 0.9$

$\therefore$ Least integer $n = 5$.

8(iii) Let Y be the number of patients, out of 200, who develop dyspepsia.

$Y \sim \mathrm{B}(200,\ 0.15)$

n is sufficiently large, $np = (200)(0.15) = 30 > 5$ and

$n(1-p) = (200)(1-0.15) = 170 > 5$

$Y \sim \mathrm{N}(30,\ 25.5)$ approx.

Using continuity correction,

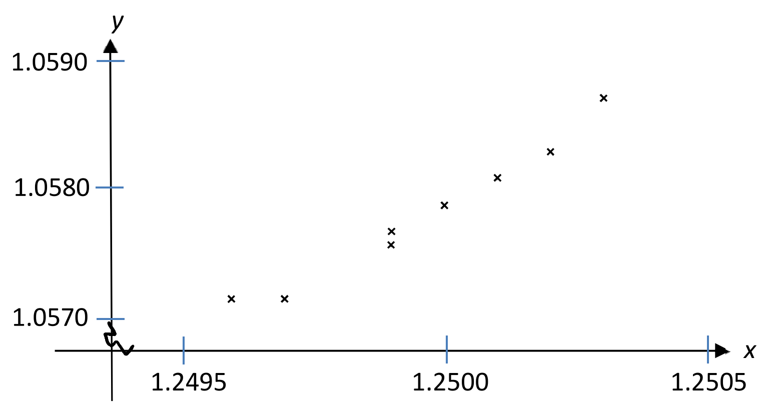
$\mathrm{P}(Y \geq 200 \times 0.15) = \mathrm{P}(Y \geq 30)$

$= \mathrm{P}(Y \geq 29.5)$

$= 0.539$ (3 s.f)

$k = 0.5$

9(i)



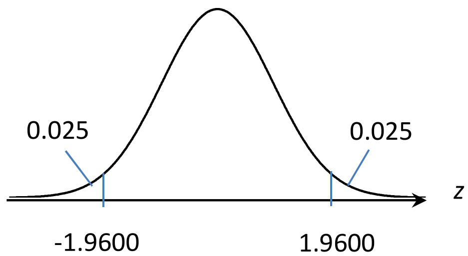
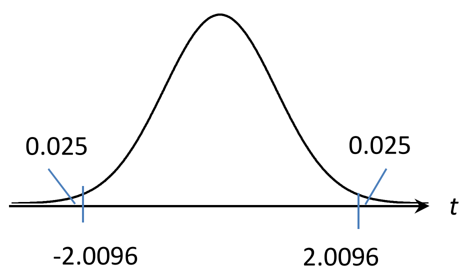
9(ii) $r = 0.9846$

The points in the scatter diagram seem to lie close to a curve rather than to a straight line. Hence a linear model may not be the best model as there could be a more appropriate model that is non-linear.

9(iii) $y = ae^{bx} + 1$ $\Rightarrow y - 1 = ae^{bx}$ $\Rightarrow \ln(y-1) = \ln a + bx$ $\Rightarrow x = \frac{1}{b}\ln(y-1) - \frac{\ln a}{b}$ From GC, the regression line of x on $\ln(y-1)$ is $x = 1.324061 + 0.025998\ln(y-1)$. $x = 1.32 + 0.0260\ln(y-1)$ When y = 1.0582, x = 1.2501	
9(iv) The points on the scatter diagram seem to lie close to a curve in which ***y* increases by increasing amounts** as x increases, suggesting that $y = ae^{bx} + 1$ may be a more appropriate model compared to $y = c + dx$. The trader can also compare the values of the product moment correlation coefficient for these 2 models, the model with a value of r closer to 1 is likely to be the better model.	
9(v) The model will not be appropriate if the rate of A$ is less than 1US$, as the model requires $y > 1$ for $\ln(y-1)$ to be defined. __**Alternative Solution**__ Since $y < 1$ is outside the range $1.0572 \le y \le 1.0587$, it is not appropriate to use the model to predict the rate of S$ when the rate of A$ is less than 1.	

10(i) Number of ways = $^{9}C_3 \times {}^{6}C_3 \times {}^{3}C_3 = 1680$	
10(ii) Number of ways = $3\times 2\times {}^{7}C_2 \times {}^{5}C_2 \times {}^{3}C_3 = 1260$ (or $3\times 2\times {}^{7}C_3 \times {}^{4}C_2 \times {}^{2}C_2 = 1260$) There are 3 ways of assigning a car to *A,* 2 ways to *B*, followed by slotting the remaining people 7 people into the 3 cars. **<u>Alternative Solution</u>** When *A* and *B* are in the same car, number of ways $= 3\times {}^{7}C_1 \times {}^{6}C_3 \times {}^{3}C_3 = 420$ (There are 3 ways of assigning a car to both *A* and *B*, followed by slotting the remaining 7 people into the 3 cars) $\therefore$ number of ways when *A* and *B* are in different cars $= 1680 - 420 = 1260$	
10(iii) Number of ways = $11! - 10!2! = 32659200$ **<u>Alternative Solution</u>** Number of ways = $9! \times {}^{10}P_2 = 32659200$ (Arrange the remaining 10 people first, then look for separated slots to accommodate *A* and *B*)	
10(iv) Number of ways when *A* and *B* are together (*B* on *A*'s left) $= 10!$ Number of ways when there is 1 person between *A* and *B* (*B* on *A*'s left) = ${}^{10}C_1 \times 9!$ Number of ways when there are 2 people between *A* and *B* (*B* on *A*'s left) = ${}^{10}C_2 \times 2! \times 8!$ $\therefore$ number of ways $= 11! - 10! - {}^{10}C_1 \times 9! - {}^{10}C_2 \times 2! \times 8! = 29030400$ **<u>Alternative Solution</u>** Number of ways = $9! \times {}^{10}C_1 \times {}^{8}C_1 = 29030400$ (Arrange the remaining 10 people first, then choose 1 slot to accommodate *A* or *B*, followed by choosing one slot to accommodate the last person, with at least 3 people between *A* and *B*)	

11(a) $y = x - 200$ Let μ g denote the population mean of Y. $\mathrm{H}_0 : \mu = 1000$ $\mathrm{H}_1 : \mu \neq 1000$ Level of significance: 5% Test Statistics : $T = \dfrac{\overline{Y} - 1000}{\dfrac{S}{\sqrt{n}}} \sim t_8$ Computation : $\overline{y} = \overline{x} - 200$ $= \dfrac{10564}{9} - 200$ $= 973.78$ unbiased estimate of population variance of Y = unbiased estimate of population variance of X $= \dfrac{1}{8}\left(12419788 - \dfrac{10564^2}{9}\right)$ $= 2499.9$ $\therefore$ p-value $= 0.15428$ Conclusion : Since p-value = 0.154 > 0.05, H_0 is not rejected at the 5% level of significance. So there is insufficient evidence to conclude that the mean mass of fruits in the basket is not 1 kg. The p-value of 0.15428 means that the smallest level of significance that can be set for this test such that the null hypothesis that the mean mass of fruits in the basket is 1 kg is rejected is 15.4%.	
11(b)(i) $\mathrm{H}_1 : \mu < 10$ Level of significance: 100α % Test Statistic : by CLT, $\overline{X} \sim N\left(10, \dfrac{2}{50}\right)$ approx $\therefore Z = \dfrac{\overline{X} - 10}{\dfrac{\sqrt{2}}{\sqrt{n}}}$ Since H_0 is rejected, $p - \text{value} = \mathrm{P}(\overline{X} < 9.56) = 0.013903 \leq \alpha$ $\therefore \alpha \geq 0.0139$	

No assumption about the distribution of X is needed. Since the sample size is large, by Central Limit Theorem, $\overline{X}$ is approximately normal.	
11(b)(ii)  Using z test, the rejection region is $\lvert z \rvert \geq 1.9600$.  Using t test, the rejection region is $\lvert t \rvert \geq 2.0096$. Since the null hypothesis was rejected under the z-test, $\dfrac{\overline{x}-10}{\sqrt{2}/\sqrt{50}} < -1.9600$ or $\dfrac{\overline{x}-10}{\sqrt{2}/\sqrt{50}} > 1.9600$. the value $\dfrac{\overline{x}-10}{\sqrt{2}/\sqrt{50}}$ can either lie inside or outside the interval $\lvert t \rvert \geq 2.0096$. Hence, it is not necessarily true that the student would have found sufficient evidence to conclude that $\mu \neq 10$.	