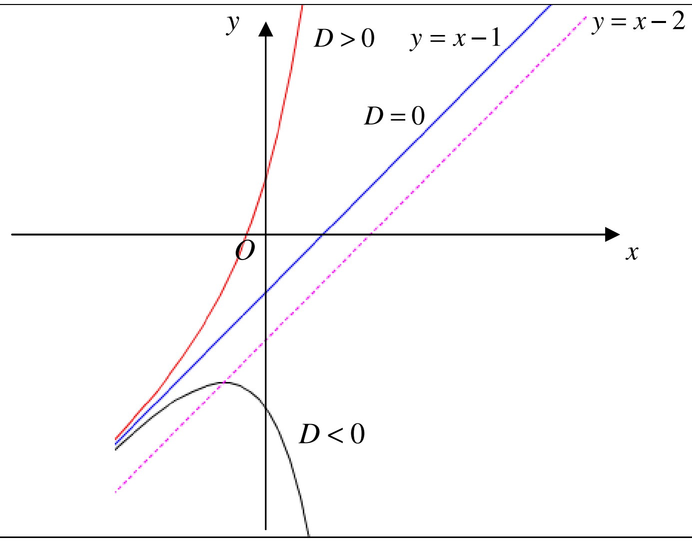
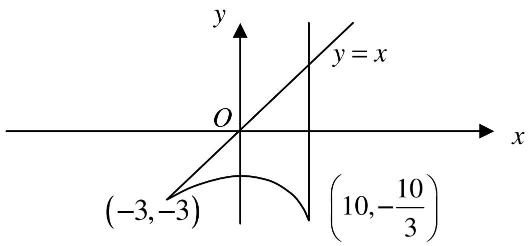
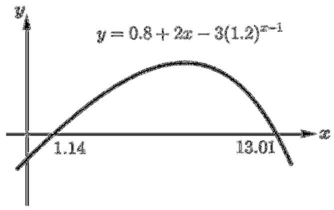
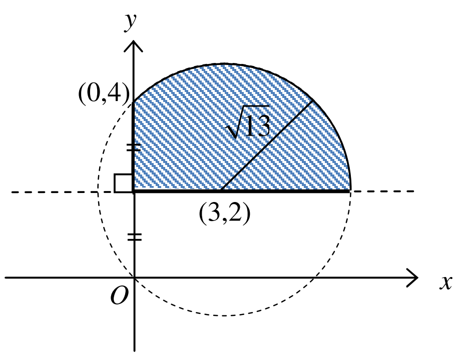
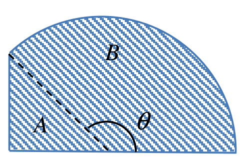
Victoria Junior College
Preliminary Examinations 2014
H2 Mathematics H2 (9740) Paper 1

	Solutions	Common Mistakes
1	$y=mx+\frac{3}{m}\Rightarrow\frac{dy}{dx}=m$ $y^2=kx\Rightarrow 2y\frac{dy}{dx}=k\Rightarrow\frac{dy}{dx}=\frac{k}{2y}$ Since the line is a tangent to the parabola for all values of m, $\frac{k}{2y}=m$ Hence $y=\frac{k}{2m}$ and $\left(\frac{k}{2m}\right)^2=kx\Rightarrow x=\frac{k}{4m^2}$ assuming $k\neq 0$ Substitute into $y=mx+\frac{3}{m}$, $\frac{k}{2m}=m\frac{k}{4m^2}+\frac{3}{m}$ $\frac{k}{4m}=\frac{3}{m}\Rightarrow k=12$ At the point of contact, $y=\frac{12}{2m}=\frac{6}{m}$ $x=\frac{12}{4m^2}=\frac{3}{m^2}$ Hence the coordinates are $\left(\frac{3}{m^2},\frac{6}{m}\right)$ <u>Alternative</u> $\left(mx+\frac{3}{m}\right)^2=kx$ $m^2x^2+6x-kx+\frac{9}{m^2}=0$ For 2 real and equal roots, $(6-k)^2-4m^2\left(\frac{9}{m^2}\right)=0$ $k=0$ (NA) or $k=12$ At the point of contact, $m^2x^2+6x-12x+\frac{9}{m^2}=0$ $m^4x^2-6m^2x+9=0$ $x=\frac{6m^2\pm\sqrt{36m^4-4(m^4)(9)}}{2m^4}$ $x=\frac{6}{2m^2}=\frac{3}{m^2}$ $y=m\left(\frac{3}{m^2}\right)+\frac{3}{m}=\frac{6}{m}$ Hence the coordinates are $\left(\frac{3}{m^2},\frac{6}{m}\right)$.	

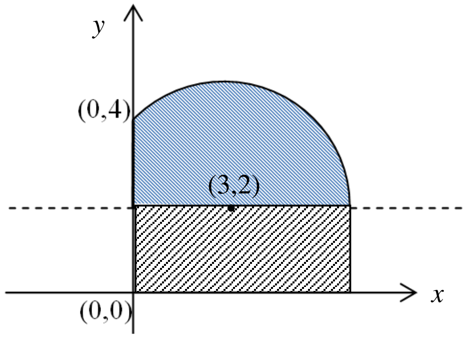
Q2	$y = x + b + De^x$ $\frac{dy}{dx} = 1 + De^x$ $= 1 + (y - x - b)$ $y - x + 2 = y - x - b + 1$ $1 - b = 2$ $b = -1$	
		
3(i)	 Area $= \int_{-3}^{10} x \, dx - \int_{-3}^{10} y_C \, dx$ $= \left[\frac{x^2}{2}\right]_{-3}^{10} - \int_0^{-0.5} \frac{t+3}{t^2-1}(8t-24)\, dt$ $= \left(50 - \frac{9}{2}\right) - 8\int_0^{-0.5} \frac{t^2-9}{t^2-1}\, dt$ $= \frac{91}{2} - 8\int_0^{-0.5} 1 - \frac{8}{t^2-1}\, dt$ $= \frac{91}{2} - 8\left[t - \frac{8}{2}\ln\left\|\frac{t-1}{t+1}\right\|\right]_0^{-0.5}$ $= \frac{91}{2} - 8(-0.5 - 4\ln 3)$ $= \frac{99}{2} + 32\ln 3$	

3(ii)	Since the x and y-coordinates are interchanged, C and D are symmetric about the line $y = x$. By Symmetry, area $= 99 + 64\ln 3$	
4	Let P_n be the proposition that $\sum_{r=1}^{n} a(2b)^r = \frac{2ab(1-(2b)^n)}{1-2b}$. Consider P_1 : $\text{LHS} = a(2b)^1 = 2ab$ $\text{RHS} = \frac{2ab(1-(2b)^1)}{1-2b} = 2ab$ $\therefore$ P_1 is true. Assume P_k is true for some $k \geq 1$, that is, $\sum_{r=1}^{k} a(2b)^r = \frac{2ab(1-(2b)^k)}{1-2b}$. Consider P_{k+1} : $\text{RHS} = \frac{2ab(1-(2b)^{k+1})}{1-2b}$ $\text{LHS} = \left[\sum_{r=1}^{k} a(2b)^r\right] + a(2b)^{k+1}$ $= \frac{2ab(1-(2b)^k)}{1-2b} + a(2b)^{k+1}$ $= \frac{2ab}{1-2b}\left[(1-(2b)^k) + (1-2b)(2b)^k\right]$ $= \frac{2ab}{1-2b}\left[1-(2b)^k + (2b)^k - (2b)^{k+1}\right]$ $= \frac{2ab(1-(2b)^{k+1})}{1-2b} = \text{RHS}$ $\therefore$ P_k is true $\Rightarrow \mathrm{P}_{k+1}$ is true Hence, (i) P_1 is true, (ii) P_k is true $\Rightarrow \mathrm{P}_{k+1}$ is true . By Mathematical induction, P_n is true for all positive integer values of n. For the series to be convergent, $\lvert 2b \rvert < 1$ $\Rightarrow \left\{ b \in \mathbb{R} : -\frac{1}{2} < b < \frac{1}{2} \right\}$ $S_\infty = \frac{2ab}{1-2b}$.	
5(i)	$w_n = 2.8 + 2(n-1) - 3(1.2)^{n-1}$ $= 0.8 + 2n - 3(1.2)^{n-1}$	

<table>
<tr><td>5(ii)</td><td>

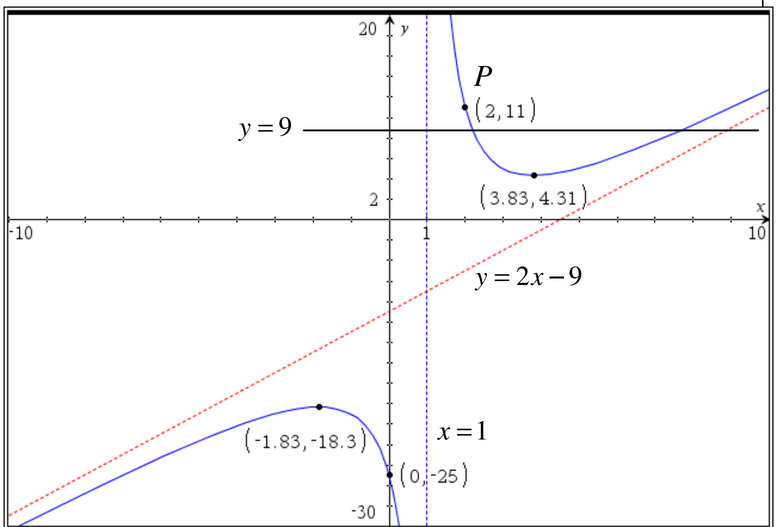
$w_n > 0$ when $1.14 < n < 13.01$
$\therefore$ set of values = $\{n \in \mathbb{Z}^+ : 2 \le n \le 13\}$.</td><td></td></tr>
<tr><td>5(iii)</td><td>$$\sum_{n=2}^{13} w_n = \sum_{n=2}^{13}\left(0.8 + 2n - 3(1.2)^{n-1}\right)$$
$$= 0.8(12) + \frac{12}{2}[4 + 26] - \frac{3(1.2)\left(1-(1.2)^{12}\right)}{1-1.2}$$
$$= 207.6 - 18(1.2)^{12}$$</td><td></td></tr>
<tr><td>6(i)</td><td>
</td><td></td></tr>
<tr><td>6(ii)</td><td>$2 \le |z| \le 2\sqrt{13}$</td><td></td></tr>
<tr><td>6(iii)</td><td>

Area of A = $\frac{1}{2}(3)(2) = 3$
Area of B = $\frac{1}{2}r^2\theta = \frac{1}{2}(13)(\pi - \tan^{-1}\frac{2}{3}) = 16.598335...$
Total area = 19.598 (3 dp)
Alternative:</td><td></td></tr>
</table>

	 Equation of the circle: $(x-3)^2+(y-2)^2=13$ Equation of upper semi-circle: $y=2+\sqrt{13-(x-3)^2}$ area $=\int_0^{3+\sqrt{13}} 2+\sqrt{13-(x-3)^2}\ \mathrm{d}x\ -2(3+\sqrt{13})=19.598$	
7(i)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x}=2\times\frac{1}{2t}=\frac{1}{t}$ At P, equation of normal to C is $y-2p=-p(x-p^2)$ $y=-px+p^3+2p$	
(ii)	When $p=\sqrt{2}$, equation of normal is $y=-\sqrt{2}x+2\sqrt{2}+2\sqrt{2}$ $y=-\sqrt{2}x+4\sqrt{2}$ When this normal meets C again, $2t=-\sqrt{2}t^2+4\sqrt{2}$ $\sqrt{2}t^2+2t-4\sqrt{2}=0$ $\left(t-\sqrt{2}\right)\left(\sqrt{2}t+4\right)=0$ $t=\sqrt{2}$ or $t=-\frac{4}{\sqrt{2}}=-2\sqrt{2}$ At P, $t=\sqrt{2}$. Hence, at Q, $t=-2\sqrt{2}$. $Q\equiv\left(8,-4\sqrt{2}\right)$ Since $P\equiv\left(2,2\sqrt{2}\right)$, the gradient of line OP is $\frac{2\sqrt{2}-0}{2-0}=\sqrt{2}$ Since $Q\equiv\left(8,-4\sqrt{2}\right)$, the gradient of line OQ is $\frac{-4\sqrt{2}-0}{8-0}=-\frac{1}{\sqrt{2}}$ Since $\sqrt{2}\times\left(-\frac{1}{\sqrt{2}}\right)=-1$, the lines OA and OB are perpendicular.	

<table>
<tr><td>8</td><td>

$$\frac{2x^2-11x+25}{x-1}=2x-9+\frac{16}{x-1}$$



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<tr><td>8(i)(a)</td><td>

The line $y = 9$ cuts the graph of $y = \mathrm{f}(x)$ more than once. Hence f is not a one-one function and f^{-1} does not exist.

Alternatively, since f(3) = f(5), f is not a one-one function. f^{-1} does not exist.

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<tr><td>8(i)(b)</td><td>

From graph, $\lambda > 3.83$, least integer $\lambda = 4$

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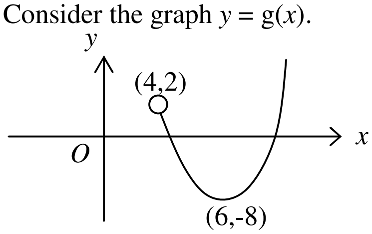
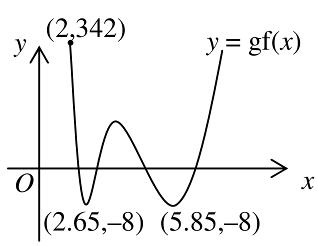
<tr><td>8(ii)</td><td>

Consider the graph $y = \mathrm{g}(x)$.



$$\mathrm{D_f}=[2,\infty)\xrightarrow{\mathrm{f}}\mathrm{R_f}=[4.31,\infty)\xrightarrow{g}\mathrm{R_{gf}}=[-8,\infty)$$

Alternatively, consider graph of gf, with $\mathrm{D_f}$ as domain.



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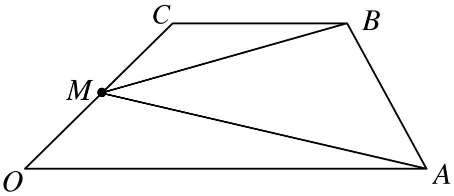
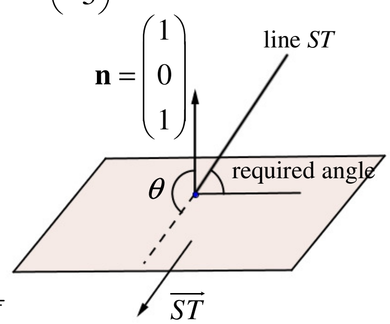
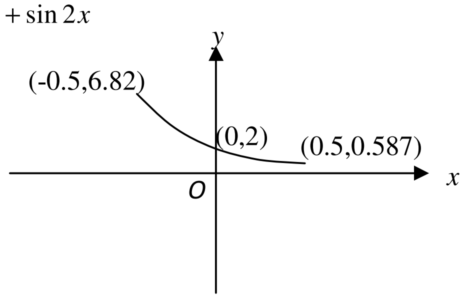
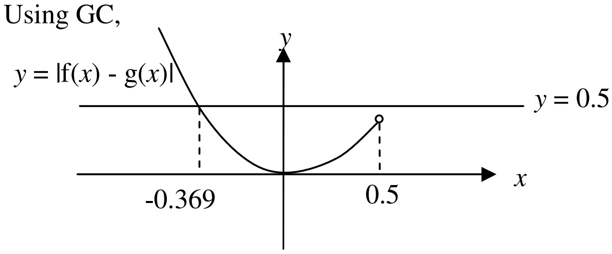
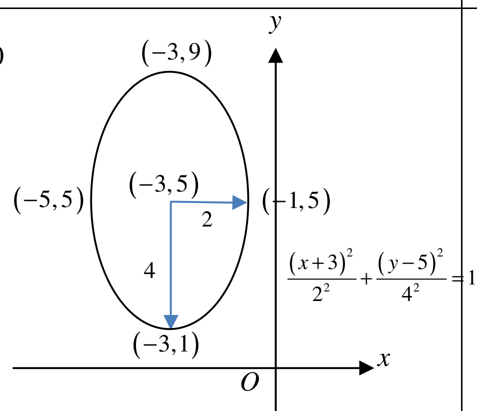
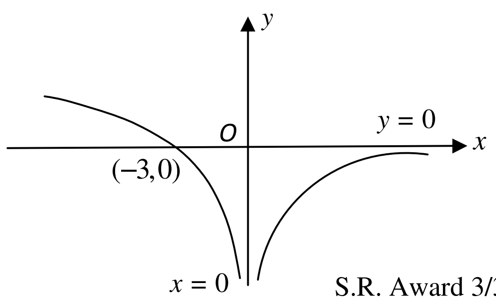
9(a)	$\overrightarrow{CB} = k\underset{\sim}{a}$ $\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} = \underset{\sim}{c} + k\underset{\sim}{a}$ $\overrightarrow{OM} = \frac{1}{2}\underset{\sim}{c}$  $\overrightarrow{MB} = \overrightarrow{OB} - \overrightarrow{OM} = \underset{\sim}{c} + k\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{c},\ \overrightarrow{MA} = \overrightarrow{OA} - \overrightarrow{OM} = \underset{\sim}{a} - \frac{1}{2}\underset{\sim}{c}$ area of triangle AMB $= \frac{1}{2}\left\|\overrightarrow{MB} \times \overrightarrow{MA}\right\| = \frac{1}{2}\left\|\left(\underset{\sim}{c} + k\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{c}\right) \underset{\sim}{\times} \left(\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{c}\right)\right\|$ $= \frac{1}{2}\left\|\left(k\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{c}\right) \times \left(\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{c}\right)\right\|$ $= \frac{1}{2}\left\|k(\underset{\sim}{a} \times \underset{\sim}{a}) - \frac{1}{2}k(\underset{\sim}{a} \times \underset{\sim}{c}) + \frac{1}{2}(\underset{\sim}{c} \times \underset{\sim}{a}) - \frac{1}{4}(\underset{\sim}{c} \times \underset{\sim}{c})\right\|$ $= \frac{1}{2}\left\|-\frac{1}{2}k(\underset{\sim}{a} \times \underset{\sim}{c}) - \frac{1}{2}(\underset{\sim}{a} \times \underset{\sim}{c})\right\| \quad (\because \underset{\sim}{a} \times \underset{\sim}{a} = \underset{\sim}{c} \times \underset{\sim}{c} = \underset{\sim}{0}, \underset{\sim}{c} \times \underset{\sim}{a} = -\underset{\sim}{a} \times \underset{\sim}{c})$ $= \frac{1}{4}(1+k)\left\|(\underset{\sim}{a} \times \underset{\sim}{c})\right\|$	
9(b)(i)	$\overrightarrow{OS} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \overrightarrow{OT} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \Rightarrow \overrightarrow{ST} = \begin{pmatrix} -1 \\ 1 \\ -5 \end{pmatrix}$ $x + z = 5 \Rightarrow \underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 5$  $\cos\theta = \frac{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ -5 \end{pmatrix}}{\sqrt{2}\sqrt{27}} = \frac{-6}{\sqrt{2}\sqrt{27}}$ $\Rightarrow \theta = 144.736^\circ$ the acute angle between line ST and the normal $= 180^\circ - \theta = 35.264^\circ$ $\therefore$ the required angle is $90^\circ - 35.264^\circ \approx 54.7^\circ$	
9(b)(ii)	Normal vector of $q = \begin{pmatrix} a \\ b \\ 1 \end{pmatrix}$ q is parallel to the line $ST \Rightarrow \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ -5 \end{pmatrix} = 0$ $\Rightarrow -a + b - 5 = 0$ $\Rightarrow b = a + 5$ ------ (1) The acute angle between p and q is 60°	

	$\cos\theta = \dfrac{\begin{pmatrix}1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}a\\b\\1\end{pmatrix}}{\sqrt{2}\sqrt{a^2+b^2+1}} = \dfrac{a+1}{\sqrt{2}\sqrt{a^2+b^2+1}}$ Since $a<-1$, $a+1$ is negative. $\dfrac{a+1}{\sqrt{2}\sqrt{a^2+b^2+1}} = \cos(180^\circ - 60^\circ) = -\dfrac{1}{2}$ $-2(a+1) = \sqrt{2}\sqrt{a^2+b^2+1}$ ------------------ (2) Sub (1) into (2) and square both sides, $4(a^2+2a+1) = 2(a^2+a^2+10a+25+1)$ $12a = -48$ $a = -4$	
10	$y = \cos^{-1}\left(\dfrac{1}{x}\right) \Rightarrow \cos y = \dfrac{1}{x}$ $x^2 = \sec^2 y$ Volume, $V = \pi\int_0^h \sec^2 y \,\mathrm{d}y$ $= \pi\left[\tan y\right]_0^h$ $= \pi\tan h$	
	$V = \pi\tan h$ $\dfrac{\mathrm{d}V}{\mathrm{d}t} = \pi\sec^2 h\dfrac{\mathrm{d}h}{\mathrm{d}t}$ $\dfrac{\mathrm{d}V}{\mathrm{d}t} = -\dfrac{m}{h}$, where m is a positive constant $\pi\sec^2 h\dfrac{\mathrm{d}h}{\mathrm{d}t} = -\dfrac{m}{h}$ $h\sec^2 h\dfrac{\mathrm{d}h}{\mathrm{d}t} = -\dfrac{m}{\pi}$ Let $m = k$ $h\sec^2 h\dfrac{\mathrm{d}h}{\mathrm{d}t} = -\dfrac{k}{\pi}$, where k is a positive constant	
	$\int h\sec^2 h \,\mathrm{d}h = \int -\dfrac{k}{\pi}\,\mathrm{d}t$ $h\tan h - \int \tan h \,\mathrm{d}h = -\dfrac{k}{\pi}t + C$ $h\tan h + \ln\lvert\cos h\rvert = -\dfrac{k}{\pi}t + C$ When $t = 0$, $h = \dfrac{\pi}{3}$, $C = \dfrac{\pi}{3}\sqrt{3} + \ln\left\lvert\dfrac{1}{2}\right\rvert = \dfrac{\sqrt{3}\pi}{3} - \ln 2$ When $t = 30$, $h = 0$	

	$30\left(-\dfrac{k}{\pi}\right)+C=0$ $k=\dfrac{C\pi}{30}=\dfrac{\pi}{30}\left(\dfrac{\sqrt{3}\pi}{3}-\ln 2\right)=0.117$ (3 sig. fig.)	
11 (i)	$y=\mathrm{f}(x)=\dfrac{2\cos 2x}{1+\sin 2x}$  	
(ii)	$\dfrac{2\cos 2x}{1+\sin 2x}=\dfrac{2\left(1-\dfrac{(2x)^2}{2}+\cdots\right)}{1+2x+\cdots}$ $=2\left(1-2x^2+\cdots\right)\left(1+2x+\cdots\right)^{-1}$ $=2\left(1-2x^2+\cdots\right)\left(1-2x+4x^2+\cdots\right)$ $=2\left(1-2x+4x^2-2x^2+\cdots\right)$ $=2-4x+4x^2+\cdots$ (a) The equation of the tangent to the curve $y=\mathrm{f}(x)$ is $y=2-4x$. (b) By Maclaurin's Theorem, $\dfrac{\mathrm{f}^{(n)}(0)}{n!}=k$. Hence $\mathrm{f}^{(n)}(0)=k(n!)$.	
(iii)	$\mathrm{g}(x)=2-4x+4x^2+\cdots$ **S**ubstitute $x=\dfrac{\pi}{12}$, we have $\dfrac{2\cos\dfrac{\pi}{6}}{1+\sin\dfrac{\pi}{6}}\approx 2-4\left(\dfrac{\pi}{12}\right)+4\left(\dfrac{\pi}{12}\right)^2$ $\dfrac{2\left(\dfrac{\sqrt{3}}{2}\right)}{1+\dfrac{1}{2}}\approx 2-\dfrac{\pi}{3}+\dfrac{\pi^2}{36}$ $\dfrac{2}{3}\left(\sqrt{3}\right)\approx 2-\dfrac{\pi}{3}+\dfrac{\pi^2}{36}$ $\sqrt{3}\approx 3-\dfrac{\pi}{2}+\dfrac{\pi^2}{24}$ The error could be reduced by including more terms with higher powers of x in the expansion.	

<table>
<tr><td>(iv)</td><td>$\left|\mathrm{f}(x)-\mathrm{g}(x)\right| \le 0.5$, $-0.5 < x < 0.5$
Using GC,


Solution set = $\{x \in \mathbb{R} : -0.369 \le x < 0.5\}$</td><td></td></tr>
<tr><td>12(a)(i)</td><td>$\left(x^2+6x+5\right)+\dfrac{(y-5)^2}{4}=0$
$(x+3)^2-4+\dfrac{(y-5)^2}{4}=0$
$(x+3)^2+\dfrac{(y-5)^2}{4}=4$
$\dfrac{(x+3)^2}{2^2}+\dfrac{(y-5)^2}{4^2}=1$

</td><td></td></tr>
<tr><td>(a)(ii)</td><td>Translation of 3 units in the positive x- direction, then stretch parallel to the x-axis, factor 2, with y-axis invariant, followed by a translation of 5 units in the negative y - direction</td><td></td></tr>
<tr><td>(a)(iii)</td><td>For C to have 2 asymptotes, it should represent a hyperbola. Hence the set of values of $k = \{k \in \mathbb{R} : k > 0\}$.</td><td></td></tr>
<tr><td>b) (i)</td><td>

S.R. Award 3/3 only if everything is correct.</td><td></td></tr>
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b) (ii)