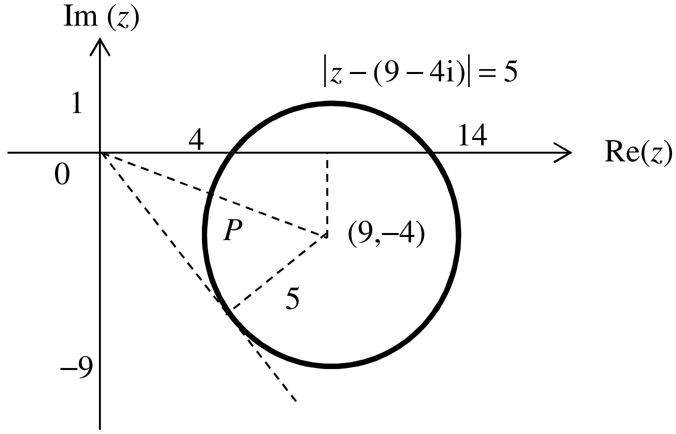
TPJC 2014 JC2 Preliminary Examination
H2 Mathematics Paper 2 Solutions
Section A: Pure Mathematics

1 (i) **(8m)**	
(ii)(a)	The smallest value of $\lvert z\rvert$ is the length of $OP = \sqrt{9^2+4^2}-5$ $= \sqrt{97}-5$
(b)	Using similar triangles, $\dfrac{x}{9}=\dfrac{\sqrt{97}-5}{\sqrt{97}} \Rightarrow x=9\left(\dfrac{\sqrt{97}-5}{\sqrt{97}}\right)$ $\dfrac{y}{4}=\dfrac{\sqrt{97}-5}{\sqrt{97}} \Rightarrow y=4\left(\dfrac{\sqrt{97}-5}{\sqrt{97}}\right)$ $z=\left(9-\dfrac{45}{\sqrt{97}}\right)-\mathrm{i}\left(4-\dfrac{20}{\sqrt{97}}\right)$
(iii)	$\lvert\arg z\rvert$ is as large as possible $\Rightarrow$ arg z is as small as possible. $\arg z = -\tan^{-1}\dfrac{4}{9}-\sin^{-1}\dfrac{5}{\sqrt{97}}$ $=-0.9507$

(iii)	Volume generated $=\pi(4)^2(2)-\ \pi\int_{-2}^{0}\left(\dfrac{-x^2+4x+12}{x+3}\right)^2\,\mathrm{d}x$ $= 34.8$ units^3 (to 3 s.f)

3(i) **(10m)**	$u = ye^t \Rightarrow \frac{du}{dt} = \frac{dy}{dt}e^t + ye^t \Rightarrow \frac{dy}{dt} = e^{-t}\frac{du}{dt} - y$ Hence, $\frac{dy}{dt} + y = 4t+3 \Rightarrow \left(e^{-t}\frac{du}{dt} - y\right) + y = 4t+3 \Rightarrow \frac{du}{dt} = (4t+3)e^t$ Integrating with respect to t, $u = \int (4t+3)e^t \, dt$ $= (4t+3)e^t - \int 4e^t \, dt$ $= (4t-1)e^t + C$ $\Rightarrow ye^t = (4t-1)e^t + C$ $\Rightarrow y = 4t - 1 + Ce^{-t}$ When $t = 0$, $y = 0.2$, $0.2 = 4\times 0 - 1 + Ce^0 \Rightarrow C = 1.2$ $\therefore y = 4t - 1 + 1.2e^{-t}$
(ii)	When $t = 1$, $y = 4\times 1 - 1 + 1.2e^{-1} = 3.44$ (to 3 s.f.) Hence, the expected number of rabbits in a decade is 3.44 million (to 3 s.f.).
(iii)	$\frac{dy}{dt} = 3 - y$ OR $\frac{dy}{dt} + y = 3$ $\frac{dy}{dt} = 3 - y \Rightarrow \int \frac{1}{3-y}\, dy = \int 1 \, dt \Rightarrow -\ln\lvert 3-y\rvert = t + C$ Hence, $\lvert 3-y\rvert = e^{-t-C} \Rightarrow y = 3 - Ae^{-t}$, $A = \pm e^{-C}$ When $t = 0$, $y = 0.2$, therefore $0.2 = 3 - Ae^0 \Rightarrow A = 2.8$ $\therefore y = 3 - 2.8e^{-t}$
(iv)	$y = 3 - 2.8e^{-t}$ As $t \to \infty, e^{-t} \to 0 \Rightarrow y \to 3$ The number of rabbits will approach 3 million in the long run.

4(i) **(13m)**	Study loan amount = $\frac{10}{100}(28000) = 2800$ Months needed to repay $\frac{2800}{100} = 28$ $\therefore$ She needs 28 months to repay her study loan.
(ii)	Tuition fee loan amount = $\frac{90}{100}(28000) = 25200$

n	Outstanding amount at the end of nth month
1	$1.004(25200 - k)$
2	$1.004^2(25200) - 1.004^2 k - 1.004k$
…	…
n	$1.004^n(25200) - 1.004^n k - 1.004^{n-1} k - ... - 1.004k$

	Oustanding amount at the end of nth month $= 1.004^n(25200) - 1.004^n k - 1.004^{n-1}k - ... - 1.004k$ $= 1.004^n(25200) - k(1.004^n + 1.004^{n-1} + ... + 1.004)$ $= 1.004^n(25200) - k\left(\frac{1.004(1.004^n - 1)}{1.004 - 1}\right)$ $= 1.004^n(25200) - 251k(1.004^n - 1)$ (Shown) When $n = 240$, $1.004^{240}(25200) - 251k(1.004^{240} - 1) = 0$ $251k(1.004^{240} - 1) = 1.004^{240}(25200)$ $k = \frac{1.004^{240}(25200)}{251(1.004^{240} - 1)}$ $k = 162.8857394$ $\therefore$ minimum monthly instalment = \$162.89
(iii)	Amount paid = $162.8857394(240) = 39092.57746$ Total interest paid = $39092.57746 - 25200$ $= 13892.57746 \approx 13892.58$
(iv)	$k = \frac{1.005^{240}(25200)}{\frac{1.005}{0.005}(1.005^{240} - 1)}$ $k = 179.6424147$ Total amount paid = $179.6424147(240) = 43114.17952$ Difference in amount paid = $43114.17952 - 39092.57746$ $= 4021.602059$ $\therefore$ Jaslyn has to pay \$4021.60 more.

Section B: Statistics

5 **(4m)**	$\mathrm{P}(2X < 1) = 0.3$ $\mathrm{P}(X < \frac{1}{2}) = 0.3$ $\mathrm{P}(Z < \frac{\frac{1}{2} - \mu}{\sigma}) = 0.3$

	$\frac{\frac{1}{2}-\mu}{\sigma}=-0.52440$.................(1) $\text{P}(-\mu < X < 3\mu)=0.96$ $\text{P}(-\frac{2\mu}{\sigma} < Z < \frac{2\mu}{\sigma})=0.96$ $\text{P}(Z < -\frac{2\mu}{\sigma})=0.02$ $-\frac{2\mu}{\sigma}=-2.0537$................(2) Solving (1) & (2), $\mu \approx 1.0218 = 1.02$ (3s.f.) $\sigma \approx 0.995076 = 0.995$ (3s.f.)

6(i) **(5m)**	List the student names in a sequential order. Sampling interval, $k=\frac{6400}{200}=32$ Randomly select the first student from 1 to 32. Subsequently, select every 32nd student until a sample of 200 names is selected. **Alternative presentation** First, obtain a list of all names of the students in the university and assign sequential numbers to all of them to establish the sampling frame. Calculate the sampling interval $k=\frac{6400}{200}=32$. Choose a random number from 1 to 32 as a starting point. (e.g. we choose 20 as the starting value). Choose 20, 52, 84, ……, until 200 students are selected for the focus group discussion. Systematic sampling does not ensure a proportionate number of students from each faculty is being represented. Students from various faculties are not well represented.
6(ii)	Stratified Sampling is used. Divide the student population into the strata, that is, the various faculties. Draw samples separately from each stratum, with sample size proportional to the relative size of the stratum. Students from each stratum are selected randomly.

	Alternative presentation Random samples of students are chosen from each faculty with sample sizes proportional to the size of the respective faculty.

7(i) **(5m)**	Let X be the number of days, out of 7, on which there will have more than 10 cm of snowfall. $X \sim \mathrm{B}(7, 0.19)$ $\mathrm{P}(X < 3) = \mathrm{P}(X \le 2)$ $= 0.86873$ (5 s.f.) $= 0.869$ (3 s.f.)
(ii)	$\mathrm{P}(X \ge 2) = 1 - \mathrm{P}(X \le 1)$ $= 1 - 0.60440$ (5 s.f.) $= 0.39560$ (5 s.f.) Let Y be the number of 7 winter day periods, out of 12, in which there will have at least 2 days with more than 10 cm of snowfall. $Y \sim \mathrm{B}(12, 0.39560)$ $\mathrm{P}(Y = 4) = 0.216$ (3 s.f.)

8(i) **(7m)**	Number of Kings $= 3+2+6+4 = 15$ Number of Queens $= 4+5+4+1 = 14$ Number of Jacks $= 1+3+2+5 = 11$ $\mathrm{P}(\text{three Kings or three Queens or three Jacks})$ $= \mathrm{P}(\text{three Kings}) + \mathrm{P}(\text{three Queens}) + \mathrm{P}(\text{three Jacks})$ $= \frac{15}{40} \times \frac{14}{39} \times \frac{13}{38} + \frac{14}{40} \times \frac{13}{39} \times \frac{12}{38} + \frac{11}{40} \times \frac{10}{39} \times \frac{9}{38}$ $= \frac{123}{1235}$
(ii)	Number of Diamonds $= 6+4+2 = 12$ Number of Spades $= 3+4+1 = 8$

	$\mathrm{P}(\text{three Kings or three Queens or three Jacks} \mid \text{two Spades and one Diamond})$ $= \dfrac{\mathrm{P}(\text{three Kings or three Queens or three Jacks} \cap \text{two Spades and one Diamond})}{\mathrm{P}(\text{two Spades and one Diamond})}$ $= \dfrac{\mathrm{P}(\text{three Kings}) + \mathrm{P}(\text{three Queens})}{\mathrm{P}(\text{two Spades and one Diamond})}$ $= \dfrac{\frac{3}{40}\times\frac{2}{39}\times\frac{6}{38}\times\frac{3!}{2!}+\frac{4}{40}\times\frac{3}{39}\times\frac{4}{38}\times\frac{3!}{2!}}{\frac{8}{40}\times\frac{7}{39}\times\frac{12}{38}\times\frac{3!}{2!}}$ $= \dfrac{1}{8}$

9(i) **(8m)**	$\bar{x} = \dfrac{371}{15} = 24.7333 = 24.7$ (3 s.f.) $s^2 = \dfrac{1}{14}\left[\sum x^2 - \dfrac{(\sum x)^2}{15}\right] = \dfrac{1}{14}\left[9425 - \dfrac{371^2}{15}\right] = 17.781 = 17.8$ (3 s.f.)
(ii)	Assuming that the breaking strength of a climbing gear follows a normal distribution. To test $\mathrm{H}_0: \mu = 23$ Against $\mathrm{H}_1: \mu > 23$ Apply T test Level of significance: 10% Reject H_0 if p-value < 0.1 Using GC, $\mu_0 = 23, \bar{x} = 24.7333, s^2 = 17.781, n = 15, t_{test} = 1.59199, p-\text{value} = 0.06685$ Since p–value = 0.06685 < 0.10, we reject H_0 and conclude that there is sufficient evidence, at 10% level of significance, that the <u>mean</u> breaking strength of the climbing gear has increased. Central Limit Theorem does not apply here since the sample size $\underline{n = 15}$ is small.

10(i) **(8m)**	

S(10)	R(14)	P(6)	No. of arrangements
1	2	4	${}^{10}C_1\,{}^{14}C_2\,{}^{6}C_4 = 13650$
1	3	3	${}^{10}C_1\,{}^{14}C_3\,{}^{6}C_3 = 72800$

	<table><tr><td>2</td><td>2</td><td>3</td><td>$^{10}C_2\,^{14}C_2\,^{6}C_3 = 81900$</td></tr></table> Total number of selections = 168350
(ii)	Number of arrangements = 2! 2! 5! = 480
(iii)	Number of arrangements = $\dfrac{4!}{4}\,^{4}C_3\,3! = 144$

11(i) **(11m)**	$r = 0.97496 = 0.975$ (3 s.f.) This $r = 0.975$ indicates that there is a strong positive linear correlation between v and $\sqrt{s}$.
(ii)	Since 0.975 is closer to 1 compared with 0.965, the regression line of $\sqrt{s}$ on v is more suitable than the regression line of s on v. Equation of regression line of $\sqrt{s}$ on v is $\sqrt{s} = 0.066336v - 0.017748$ i.e. $\sqrt{s} = 0.0663v - 0.0177$ (3 s.f.)
(iii)	(a) When $v = 0$, $\sqrt{s} = -0.0177 \neq 0$ suggests that there is an error in the data. (b) For each additional km/h that is added to the speed, an additional 0.0663m is added on average to the square root of the distance travelled.
(iv)	No. 70 km/h lies outside the given data range of v. Hence the prediction is not reliable. Extrapolation is not a good practice.

12(i) **(12m)**	Let W be the number of whales spotted per day during a cruise. $W \sim \text{Po}(1.2)$ $P(W \geq 3) = 1 - P(W \leq 2)$ $= 1 - 0.87949$ $= 0.121$ (3 s.f.)
(ii)	Let X be the total number of whales spotted in n days. $X \sim \text{Po}(1.2n)$ $P(X = 0) \leq 0.05$ $e^{-1.2n} \leq 0.05$ $n \geq 2.50$ (3 s.f.)

	Hence, least $n = 3$
(iii)	Let C be the number of cruises, out of 52, that spots less than 20 whales or dolphins altogether. $C \sim \mathrm{B}(52, 1-0.96)$ $\quad C \sim \mathrm{B}(52, 0.04)$ Since $n = 52 > 50$ is large, $\qquad np = 52 \times 0.04 = 2.08 < 5$ $C \sim \mathrm{Po}(2.08)$ approximately $\mathrm{P}(C > 4) = 1 - \mathrm{P}(C \le 4)$ $\qquad\qquad = 1 - 0.93984$ $\qquad\qquad = 0.0602$ (3 s.f.)
(iv)	Let R and D be the number of whales and dolphins Stella spots in a year. $R \sim \mathrm{Po}(100 \times 1.2)$ $\qquad\qquad D \sim \mathrm{Po}(100 \times 4.5)$ $R \sim \mathrm{Po}(120)$ $\qquad\qquad D \sim \mathrm{Po}(450)$ Since $\lambda_R = 120 > 10$ and $\lambda_D = 450 > 10$, $R \sim \mathrm{N}(120, 120)$ approx. $\quad D \sim \mathrm{N}(450, 450)$ approx. $D - 4R \sim \mathrm{N}(-30, 2370)$ $\mathrm{P}(D \ge 4R) = \mathrm{P}(D - 4R \ge 0)$ $\qquad\qquad \overset{c.c}{=} \mathrm{P}(D - 4R > -0.5)$ $\qquad\qquad = 0.27226 \approx 0.272$