Answer all questions [100 marks].

1 A cubic polynomial has turning points at $A(-1, 13)$ and $B(2, -14)$.

(i) Find the equation of this polynomial. [3]

(ii) Hence find the coordinates of the point C on the graph of this polynomial such that AC is parallel to the x-axis. [2]

Suggested Solutions

Let $y = ax^3 + bx^2 + cx + d$

So, $\dfrac{dy}{dx} = 3ax^2 + 2bx + c$

Using $A(-1, 13)$,

$-a + b - c + d = 13$ --- (1)

$3a - 2b + c = 0$ --- (2)

Using $B(2, -14)$,

$8a + 4b + 2c + d = -14$ --- (3)

$12a + 4b + c = 0$ --- (4)

Using GC, $a = 2,\ b = -3,\ c = -12,\ d = 6$

Hence, $y = 2x^3 - 3x^2 - 12x + 6$

Since AC is parallel to the x-axis, y-coordinate at C is 13.

So, $13 = 2x^3 - 3x^2 - 12x + 6$

Solving, we get $x = 3.5$ or $x = -1$ (N.A.)

Therefore, the coordinates of C is (3.5, 13).

2 SRBank introduces the UniSave Bank Account to encourage young parents to save up for their child's university education. This account consists of two independent components – Ordinary Account and Birthday Account. The bank will provide an interest rate of 1% of the total amount in the Ordinary Account at the end of every year. As a bonus, the bank will deposit an amount equivalent to ten times the age of the child into the Birthday Account on the child's birthday each year, with the last deposit on the 18th birthday.

Mr and Mrs Hon intend to save up for their child's university education by depositing a fixed amount of \$3000 into the UniSave's Ordinary Account at the start of every year from the year their child turns one year old.

(i) Show that the total amount in the UniSave Bank Account at the end of the year when their child is n years old, where $n \geq 19$, is given by

$\$303000\left(1.01^n\right) - 301290$. [3]

(ii) Given that $n \geq 19$, find the least n such that the total amount in the UniSave Bank account will exceed \$70000. [2]

Suggested Solution

(i)

End of Year	Total Amount in Ordinary Account	Total Amount in Birthday Account
1	$\$3000(1.01)$	\$10
2	$\$\left[3000+3000(1.01)\right](1.01)$ $=\$3000(1.01)+3000\left(1.01^2\right)$	$\$(10+20)$
3	$\$\left[3000+3000(1.01)+3000\left(1.01^2\right)\right](1.01)$ $=\$3000(1.01)+3000\left(1.01^2\right)+3000\left(1.01^3\right)$	$\$(10+20+30)$
…	…	…
17	$\$3000\left(1.01+1.01^2+...+1.01^{17}\right)$	$\$(10+...+170)$
18	$\$3000\left(1.01+1.01^2+...+1.01^{18}\right)$	$\$(10+...+180)$
19	$\$3000\left(1.01+1.01^2+...+1.01^{19}\right)$	$\$(10+...+180)$

Total amount at the end of Year *n* (where $n \geq 19$)

$= \$3000(1.01+1.01^2+...+1.01^n)+10+20+...+180$

$= \$3000\left[\dfrac{1.01\left(1.01^n-1\right)}{1.01-1}\right]+\dfrac{18}{2}(10+180)$

$= \$303000\left(1.01^n-1\right)+1710$

$= \$303000\left(1.01^n\right)-301290$

(ii)

$303000\left(1.01^n\right)-301290>70000$

$1.01^n>1.22538$

$n>20.43$

Least $n = 21$

3 A sequence of real numbers $u_1, u_2, u_3, \ldots$ satisfies the recurrence relation

$$u_{n+1}=\frac{u_n}{u_n+3}\ , \ n\in\mathbb{Z}^+ .$$

Given that $u_1=1$, and by considering $\dfrac{2}{u_n}+1$ for $n=1,2,3$, make a suitable conjecture for u_n in the form of $\dfrac{k}{a^n-b}$, where $a,b,k\in\mathbb{Z}^+$. [2]

Prove the conjecture by Mathematical Induction. [4]

Solution

$u_1 = 1$ $\qquad$ $\dfrac{2}{u_1}+1 = \dfrac{2}{(1)}+1 = 3^1$

$u_2 = \dfrac{u_1}{u_1+3} = \dfrac{(1)}{(1)+3} = \dfrac{1}{4}$ $\qquad$ $\dfrac{2}{u_2}+1 = \dfrac{2}{\left(\frac{1}{4}\right)}+1 = 3^2$

$u_3 = \dfrac{u_2}{u_2+3} = \dfrac{\left(\frac{1}{4}\right)}{\left(\frac{1}{4}\right)+3} = \dfrac{1}{13}$ $\qquad$ $\dfrac{2}{u_3}+1 = \dfrac{2}{\left(\frac{1}{13}\right)}+1 = 3^3$

Conjecture: $u_n = \dfrac{2}{3^n - 1}$

Let P_n be the statement "$u_n = \dfrac{2}{3^n-1}$, $\forall n \in \mathbb{Z}^+$".

When $n = 1$,

LHS = $u_1 = 1$ $\qquad$ RHS = $\dfrac{2}{3^1-1} = 1 = \text{LHS}$

Hence P_1 is true.

Assume P_k is true for some $k \in \mathbb{Z}^+$ i.e $u_k = \dfrac{2}{3^k-1}$

To show that P_{k+1} is true, i.e $u_{k+1} = \dfrac{2}{3^{k+1}-1}$

$$
\begin{aligned}
\text{LHS} = u_{k+1} &= \frac{u_k}{u_k+3} \\
&= \frac{\left(\frac{2}{3^k-1}\right)}{\left(\frac{2}{3^k-1}\right)+3} \\
&= \frac{\left(\frac{2}{3^k-1}\right)}{\left(\frac{2}{3^k-1}\right)+3} \times \frac{3^k-1}{3^k-1} \\
&= \frac{2}{2+3\left(3^k-1\right)} \\
&= \frac{2}{3\left(3^k\right)-1} \\
&= \frac{2}{3^{k+1}-1} \\
&= \text{RHS}
\end{aligned}
$$

Therefore, P_{k+1} is true when P_k is true.

Since P_1 is true and P_{k+1} is true when P_k is true, by Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$.

4 **(a)** A graph with equation $y = \mathrm{f}(x)$ undergoes in succession, the following transformations:

A : A translation of 3 units in the direction of the negative x-axis

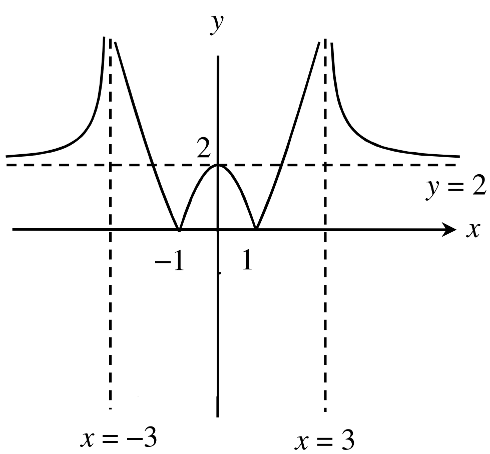
B : A reflection about the y-axis

C : A scaling parallel to the x-axis by a factor of 2

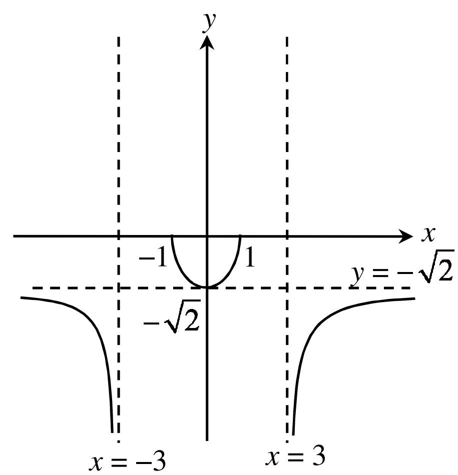
The equation of the resulting curve is given by $y = \dfrac{x-4}{2x-18}$.

Find the equation $y = \mathrm{f}(x)$. [3]

(b) The graphs of $y = |\mathrm{g}(x)|$ and $y = -\sqrt{\mathrm{g}(x)}$ are shown below.



Graph of $y = |\mathrm{g}(x)|$



Graph of $y = -\sqrt{\mathrm{g}(x)}$

Sketch the graph of $y = \mathrm{g}(x)$, showing clearly any equations of asymptote and intercepts with the axes. [3]

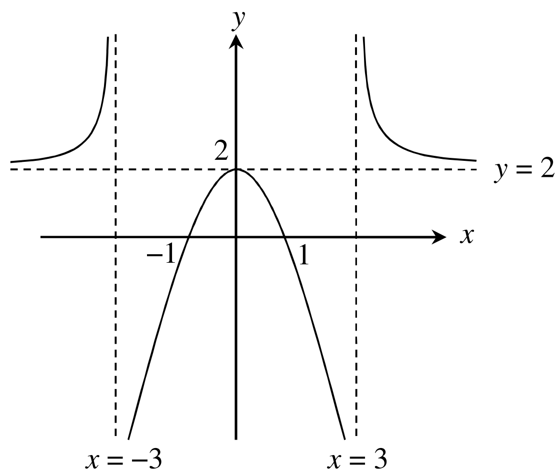
Solution

$$y = \frac{x-4}{2x-18} \xrightarrow{C'} y = \frac{2x-4}{4x-18} = \frac{x-2}{2x-9}$$

$$y = \frac{x-2}{2x-9} \xrightarrow{B'} y = \frac{-x-2}{-2x-9} = \frac{x+2}{2x+9}$$

$$y = \frac{x+2}{2x+9} \xrightarrow{A'} y = \frac{(x-3)+2}{2(x-3)+9} = \frac{x-1}{2x+3}$$

$y = \mathrm{g}(x)$



5 The curve C has equation given by $y = \dfrac{x^2+x+1}{x+1}$, $x \in \mathbb{R}$, $x \neq -1$.

(i) Without using a calculator, find the set of values that y can take. [3]

(ii) Sketch the graph of C, indicating clearly the equations of any asymptotes and the coordinates of any turning points of the curve. [2]

(iii) Show that $(-1,-1)$ lies on $y = kx + k - 1$, where $k \in \mathbb{R}$.

Hence, find the range of values of k where $kx - 1 = \dfrac{(x-1)^2 + x}{x}$ has two real roots. [2]

Suggested Solution

(i) Consider any horizontal line $y = k$, $k \in \mathbb{R}$.

Consider the intersection of the graphs $y = \dfrac{x^2+x+1}{x+1}$ and $y = k$,

i.e. $\dfrac{x^2+x+1}{x+1} = k$

$\Rightarrow x^2 + x + 1 = k(x+1)$

$\Rightarrow x^2 + x(1-k) + (1-k) = 0$

For the equation to have real solutions, we need Discriminant ≥ 0

$\Rightarrow (1-k)^2 - 4(1)(1-k) \geq 0$

$\Rightarrow k^2 + 2k - 3 \geq 0$

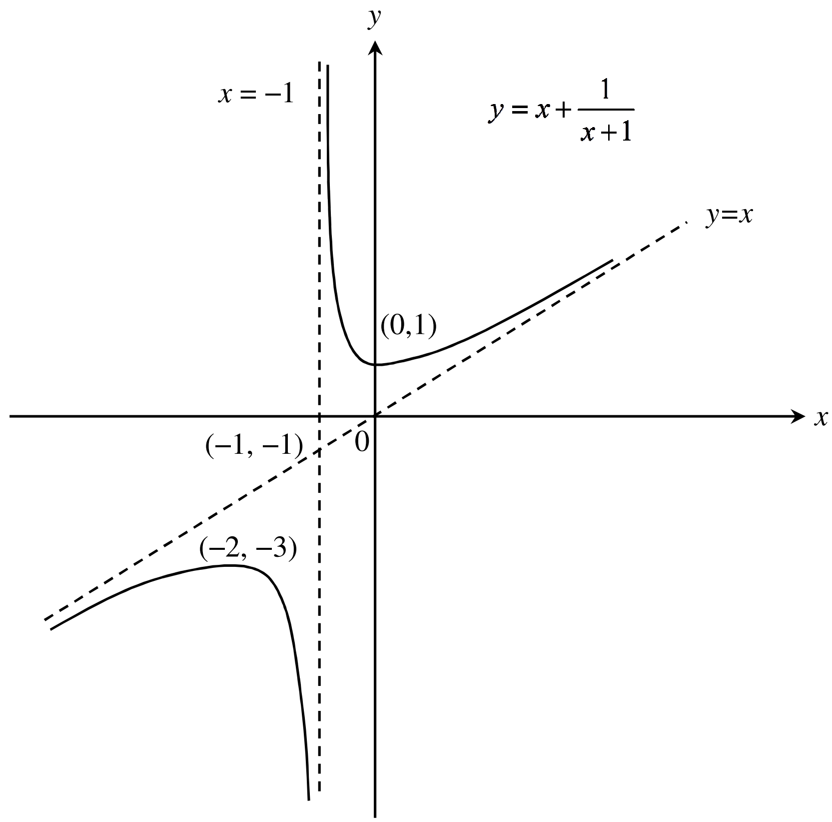
$\Rightarrow (k-1)(k+3) \geq 0$

We have $k \leq -3$ or $k \geq 1$

$\therefore$ The set of values that y can take is $\{\, y \in \mathbb{R} : y \leq -3 \text{ or } y \geq 1 \}$

$y = \dfrac{x^2+x+1}{x+1} = x + \dfrac{1}{x+1}$

(ii)



When $x=-1$, $y=k(-1)+k-1=-1$ (Shown)

To find the range of values of k where $kx-1=\dfrac{(x-1)^2+x}{x}$ has 2 real roots is the same as finding the range of values of k for which $kx+k-1=\dfrac{x^2+x+1}{x+1}$ has two real roots. So from above, is $k>1$.

6 The number of people infected with virus A is x. The rate at which x is varying at any time t is proportional to the difference between the number of infected people and the number of death due to virus A. At time t, it is also known that the number of deaths due to virus A is proportional to the square of the number of people infected with virus A. Initially there were 10 people infected and the number infected remains constant when it reaches 100.

Show that $\dfrac{dx}{dt}=\dfrac{k}{100}\left(100x-x^2\right)$, where k is a constant. Hence find x in terms of k and t in the form $x=\dfrac{p}{1+qe^{-kt}}$ where p and q are constants to be determined. [7]

Solution

Let D represent the number of deaths due to virus A.

$$\frac{\mathrm{d}x}{\mathrm{d}t} \propto (x-D)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k(x-D)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k\left(x-Ax^2\right)$$

When $x=100, \frac{\mathrm{d}x}{\mathrm{d}t}=0$

$$0 = k\left(100 - A(100)^2\right)$$

$$A = \frac{1}{100}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k\left(x - \frac{1}{100}x^2\right) = \frac{k}{100}\left(100x - x^2\right), \text{ shown}$$

Integrating $\Rightarrow \int \frac{1}{100x-x^2}\,\mathrm{d}x = \int \frac{k}{100}\,\mathrm{d}t$

$$\int \frac{1}{(100-x)x}\,\mathrm{d}x = \int \frac{k}{100}\,\mathrm{d}t$$

$$\frac{1}{100}\int \frac{1}{x} + \frac{1}{100-x}\,\mathrm{d}x = \int \frac{k}{100}\,\mathrm{d}t$$

$$\int \frac{1}{x} + \frac{1}{100-x}\,\mathrm{d}x = \int k\,\mathrm{d}t$$

$$\ln|x| - \ln|100-x| = kt + \mathrm{C}$$

$$\ln\left|\frac{x}{100-x}\right| = kt + \mathrm{C}$$

$$\frac{x}{100-x} = B\mathrm{e}^{kt} \text{ where } B = \pm\mathrm{e}^{C}$$

$$x = 100B\mathrm{e}^{kt} - Bx\mathrm{e}^{kt}$$

$$x + Bx\mathrm{e}^{kt} = 100B\mathrm{e}^{kt}$$

$$x\left(1 + B\mathrm{e}^{kt}\right) = 100B\mathrm{e}^{kt}$$

$$x = \frac{100B\mathrm{e}^{kt}}{1 + B\mathrm{e}^{kt}}$$

When $t = 0, x = 10$

$$10 = \frac{100B}{1+B}$$

$$10 + 10B = 100B$$

$$90B = 10$$

$$B = \frac{1}{9}$$

$$\therefore x = \frac{100\left(\frac{1}{9}\right)\mathrm{e}^{kt}}{1+\left(\frac{1}{9}\right)\mathrm{e}^{kt}}$$

$$= \frac{100\mathrm{e}^{kt}}{9+\mathrm{e}^{kt}}$$

$$= \frac{100}{\left(\frac{9+\mathrm{e}^{kt}}{\mathrm{e}^{kt}}\right)} = \frac{100}{1+9\mathrm{e}^{-kt}}$$, where p = 100 and q = 9

7 The equations of three planes p_1, p_2, p_3 are

$$x + y + z = 3,$$

$$x - z = 3,$$

$$3x + \lambda y - 2z = \mu,$$

respectively, where λ and μ are constants.
A line l passes through the origin and the point, $A(1, 2, -1)$.

(i) Find the coordinates of B, the point of intersection between the line l and plane p_1. [3]

(ii) Find the sine of the acute angle between the line l and the plane p_1. Hence find the exact shortest distance from point A to the plane p_1. [2]

(iii) Find the conditions satisfied by λ and μ given that there is no point in common among the three planes. [3]

Solution

(i) Since B lies on the line l,

$$\mathbf{b} = \begin{pmatrix} \lambda \\ 2\lambda \\ -\lambda \end{pmatrix} \text{ for some } \lambda \in \mathbb{R}$$

$$\begin{pmatrix} \lambda \\ 2\lambda \\ -\lambda \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 3$$

$$\lambda = \frac{3}{2}$$

Coordinates of B are $\left(\frac{3}{2}, \quad 3, \quad -\frac{3}{2}\right)$

(ii) Let θ be the acute angle between the line l and plane p_1.

$$\sin\theta = \frac{\begin{pmatrix}1\\2\\-1\end{pmatrix}\bullet\begin{pmatrix}1\\1\\1\end{pmatrix}}{\left|\begin{pmatrix}1\\2\\-1\end{pmatrix}\right|\left|\begin{pmatrix}1\\1\\1\end{pmatrix}\right|}$$

$$\sin\theta = \frac{1+2-1}{\sqrt{6}\sqrt{3}} = \frac{2}{3\sqrt{2}}$$

Shortest dist = $\left|\overrightarrow{BA}\right|\sin\theta = \frac{1}{2}\left|\begin{pmatrix}-1\\-2\\1\end{pmatrix}\right|\left(\frac{2}{3\sqrt{2}}\right) = \frac{\sqrt{3}}{3}$ units

(iii) $\begin{pmatrix}1\\1\\1\end{pmatrix}\times\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}-1\\2\\-1\end{pmatrix}$

$$\begin{pmatrix}-1\\2\\-1\end{pmatrix}\bullet\begin{pmatrix}3\\\lambda\\-2\end{pmatrix} = 0$$

$$-3+2\lambda+2 = 0$$

$$\lambda = \frac{1}{2}$$

Using GC, (3, 0, 0) lies on the line of intersection between planes p_1 and p_2.

$$\therefore \begin{pmatrix}3\\0\\0\end{pmatrix}\bullet\begin{pmatrix}3\\\frac{1}{2}\\-2\end{pmatrix} \neq \mu$$

$$\mu \neq 9$$

8 A curve has the parametric equations

$$x = \cos^2 t, \; y = \sin^3 t, \text{ for } 0 \le t \le \frac{\pi}{2}.$$

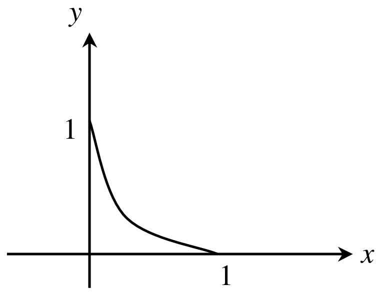
(i) Sketch the curve, indicating clearly the axial intercepts. [1]

(ii) Find the equations of the tangent and normal to the curve at the point $P(\cos^2\theta, \sin^3\theta)$, where $0 < \theta < \frac{\pi}{2}$. [4]

(iii) The tangent to the curve at P meets the x-axis at A and the normal to the curve at P meets the x-axis at B respectively. Show that the area of triangle $PBA = \frac{1}{12}\sin^5\theta\,(4 + 9\sin^2\theta)$. [3]

Solution

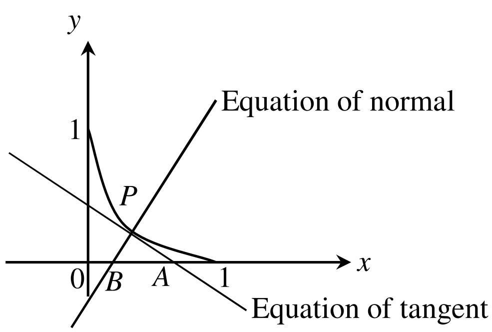
(i)



(ii) $x = \cos^2 t$, $\dfrac{dx}{dt} = -2\cos t \sin t$

$y = \sin^3 t$, $\dfrac{dy}{dt} = 3\sin^2 t \cos t$

$$\frac{dy}{dx} = \frac{3\sin^2 t \cos t}{-2\cos t \sin t} = -\frac{3}{2}\sin t$$



At the point P $(\cos^2\theta, \sin^3\theta)$, $t = \theta$

Gradient of tangent, $m = \dfrac{dy}{dx} = -\dfrac{3}{2}\sin\theta$

Equation of tangent at P $(\cos^2\theta, \sin^3\theta)$:

$$y - \sin^3\theta = -\frac{3}{2}\sin\theta\,(x - \cos^2\theta)$$

$$y = -\frac{3}{2}\sin\theta\,(x - \cos^2\theta) + \sin^3\theta$$

At the point P $(\cos^2\theta, \sin^3\theta)$, $t = \theta$

Gradient of normal, $-\dfrac{1}{m} = -\left(\dfrac{1}{-\frac{3}{2}\sin\theta}\right) = \dfrac{2}{3\sin\theta}$

Equation of normal at P $(\cos^2\theta, \sin^3\theta)$:

$$y - \sin^3\theta = \frac{2}{3\sin\theta}(x - \cos^2\theta)$$

$$y = \frac{2}{3\sin\theta}(x - \cos^2\theta) + \sin^3\theta$$

(iii) At A, When $y = 0$, $\quad 0 - \sin^3\theta = -\dfrac{3}{2}\sin\theta\,(x - \cos^2\theta)$

$$\frac{2}{3}\sin^2\theta = x - \cos^2\theta$$

$$x = \cos^2\theta + \frac{2}{3}\sin^2\theta$$

At B, When $y = 0$, $\quad 0 - \sin^3\theta = \dfrac{2}{3\sin\theta}(x - \cos^2\theta)$

$$-\frac{3}{2}\sin^4\theta = x - \cos^2\theta$$

$$x = \cos^2\theta - \frac{3}{2}\sin^4\theta$$

Area of triangle PBA

$= \dfrac{1}{2}\,[(\cos^2\theta + \dfrac{2}{3}\sin^2\theta) - (\cos^2\theta - \dfrac{3}{2}\sin^4\theta)][\sin^3\theta]$

$= \dfrac{1}{2}\,[\,\dfrac{2}{3}\sin^2\theta + \dfrac{3}{2}\sin^4\theta\,][\sin^3\theta]$

$= \dfrac{1}{12}\sin^5\theta\,(4 + 9\sin^2\theta)$ (Shown)

9 The functions f and g are defined as follows:

$$\mathrm{f} : x \mapsto \left|4x - x^2\right|, \quad x < k$$

$$\mathrm{g} : x \mapsto \sqrt{x+4}, \quad x > -4.$$

(i) State the largest value of k for the inverse function f to exist. Hence, find f^{-1} in similar form. [4]

(ii) Using the value of k found in (i), explain why the composite function gf exists. State the range of gf. Find the rule of gf in the form $bx + a$, where $a, b \in \mathbb{R}$, stating clearly its domain. [5]

Solution

(i) largest $k = 0$.

When $k = 0$,

$y = -\left(4x - x^2\right), \quad x < 0$

$y = (x-2)^2 - 4$

$x = 2 \pm \sqrt{y+4}$

$x = 2 - \sqrt{y+4} \quad (\because x < 0)$

$\mathrm{f}^{-1} : x \mapsto 2 - \sqrt{x+4}, \quad x > 0$

(ii) $\mathrm{R_f} = (0, \infty)$

$\mathrm{D_g} = (-4, \infty)$

Since $\mathrm{R_f} \subseteq \mathrm{D_g}$, so gf exists.

$(-\infty, 0) \xrightarrow[\mathrm{f}]{} (0, \infty) \xrightarrow[\mathrm{g}]{} (2, \infty)$

So $\mathrm{R_{gf}} = (2, \infty)$

$\mathrm{gf}(x) = \mathrm{g}\left(x^2 - 4x\right)$ $(\because x < 0)$

$= \sqrt{x^2 - 4x + 4}$

$= \sqrt{(x-2)^2}$

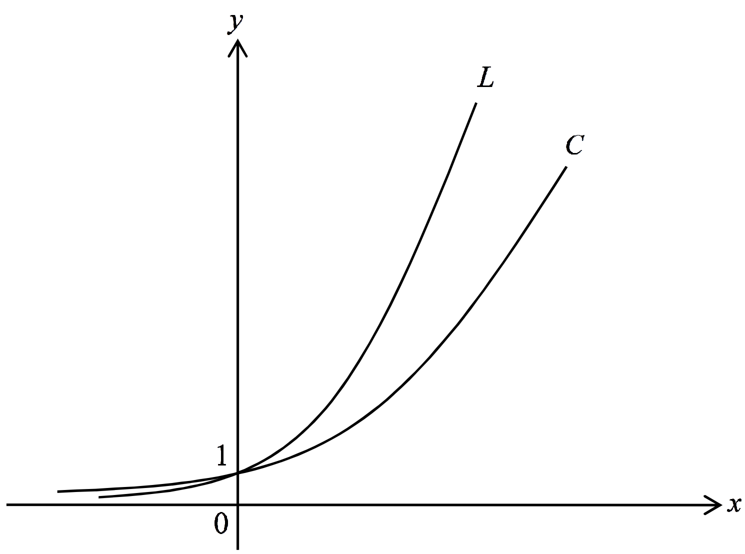
$= 2 - x$, $\left(\because \mathrm{D}_{\mathrm{gf}} = (-\infty, 0)\right)$

$\mathrm{D}_{\mathrm{gf}} = (-\infty, 0)$

10 **(a)** The curve C is defined by the parametric equations

$$x = \ln t, \quad y = \frac{t^3 + t}{t+1} \qquad \text{where } t > 0.$$

Another curve L is defined by the equation $y = \mathrm{e}^{2x}$. The graphs of C and L are shown in the diagram below.



Find the exact area of the region bounded by C, L and the line $x = \ln 2$, giving your answer in the form $\ln b$ where b is a constant to be determined. [5]

(b) The curves V and W have equations $2y = (x-1)^2 + 4$ and $y = 2x^2$ respectively. The region in the first quadrant enclosed by the curves and the y-axis is denoted by S.
Find the exact volume of the solid generated when the region S is rotated through 2π radians about the y-axis. [4]

Solution

(a) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{1}{t}$

Area of the region = $\displaystyle\int_0^{\ln 2} \mathrm{e}^{2x}\,\mathrm{d}x - \int_0^{\ln 2} y\,\mathrm{d}x$

$= \displaystyle\int_0^{\ln 2} \mathrm{e}^{2x}\,\mathrm{d}x - \int_1^{2} \left(\frac{t^3 + t}{t+1}\right)\left(\frac{1}{t}\right)\mathrm{d}t$

$$=\left[\frac{1}{2}e^{2x}\right]_0^{\ln 2}-\int_1^2 \frac{t^2+1}{t+1}\,dt$$

$$=\left[\frac{1}{2}e^{2x}\right]_0^{\ln 2}-\int_1^2 t-1+\frac{2}{t+1}\,dt$$

$$=\left[\frac{1}{2}e^{2x}\right]_0^{\ln 2}-\left[\frac{t^2}{2}-t+2\ln(t+1)\right]_1^2$$

$$=\left(\frac{1}{2}e^{2\ln 2}-\frac{1}{2}\right)-\left[\left(\frac{2^2}{2}-2+2\ln(2+1)\right)-\left(\frac{1^2}{2}-1+2\ln(1+1)\right)\right]$$

$$=\left(2-\frac{1}{2}\right)-\left[(2-2+2\ln 3)-\left(\frac{1}{2}-1+2\ln 2\right)\right]$$

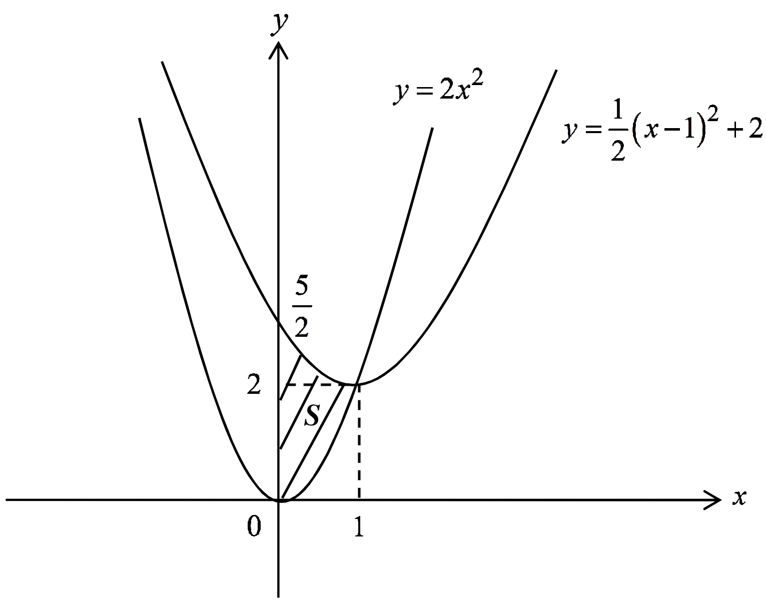
$$=\frac{3}{2}-2\ln 3-\frac{1}{2}+2\ln 2$$

$$=1+2\ln\frac{2}{3}$$

$$=\ln e+\ln\frac{4}{9}$$

$$=\ln\left(\frac{4e}{9}\right) \text{ where } b=\frac{4e}{9}$$

(b)



$$\text{Required Volume}=\pi\left[\int_0^2 \frac{y}{2}\,dy+\int_2^{\frac{5}{2}}\left(1-\sqrt{2y-4}\right)^2 dy\right]$$

$$=\pi\left\{\left[\frac{y^2}{4}\right]_0^2+\int_2^{\frac{5}{2}}\left(1-2\sqrt{2y-4}+2y-4\right)dy\right\}$$

$$= \pi + \pi \int_{2}^{\frac{5}{2}} \left(-2\sqrt{2y-4} + 2y - 3\right) \mathrm{d}y$$

$$= \pi + \pi \left[\frac{-2(2y-4)^{\frac{3}{2}}}{2\left(\frac{3}{2}\right)} + y^2 - 3y \right]_{2}^{\frac{5}{2}}$$

$$= \pi + \pi \left[-\frac{2}{3}(2y-4)^{\frac{3}{2}} + y^2 - 3y \right]_{2}^{\frac{5}{2}}$$

$$= \pi + \pi \left[\left(-\frac{2}{3}\left(2\left(\frac{5}{2}\right) - 4\right)^{\frac{3}{2}} + \left(\frac{5}{2}\right)^2 - 3\left(\frac{5}{2}\right) \right) - \left(-\frac{2}{3}(2(2)-4)^{\frac{3}{2}} + (2)^2 - 3(2) \right) \right]$$

$$= \pi + \frac{\pi}{12}$$

$$= \frac{13\pi}{12} \text{ cubic units}$$

11 It is given that $\dfrac{1-N}{(N+1)(N+2)(N+3)} = \dfrac{1}{N+1} - \dfrac{3}{N+2} + \dfrac{2}{N+3}$.

(i) Find S_n in terms of n, where $S_n = \displaystyle\sum_{r=1}^{n} \frac{4r-4}{(r+1)(r+2)(r+3)}$. [3]

(ii) Hence find S_∞, stating clearly the reason. [2]

(iii) Using the result in part (i),

(a) find $\displaystyle\sum_{r=1}^{n} \frac{4r}{(r+2)(r+3)(r+4)}$ in terms of n, [2]

(b) deduce that $\displaystyle\sum_{r=11}^{n} \frac{r-1}{(r+2)^3} < \frac{11}{156}$. [3]

Solution

(i) $S_n = \displaystyle\sum_{r=1}^{n} \frac{4r-4}{(r+1)(r+2)(r+3)}$

$$= -4\sum_{r=1}^{n} \frac{1-r}{(r+1)(r+2)(r+3)}$$

$$= -4\sum_{r=1}^{n} \left[\frac{1}{r+1} - \frac{3}{r+2} + \frac{2}{r+3} \right]$$

$$=-4\begin{pmatrix}(\frac{1}{2}-\frac{3}{3}+\frac{2}{4})\\+(\frac{1}{3}-\frac{3}{4}+\frac{2}{5})\\+(\frac{1}{4}-\frac{3}{5}+\frac{2}{6})\\+(\frac{1}{5}-\frac{3}{6}+\frac{2}{7})\\+\ldots\ldots\ldots\ldots\\+(\frac{1}{n-1}-\frac{3}{n}+\frac{2}{n+1})\\+(\frac{1}{n}-\frac{3}{n+1}+\frac{2}{n+2})\\+(\frac{1}{n+1}-\frac{3}{n+2}+\frac{2}{n+3})\end{pmatrix}$$

$$S_n=\frac{2}{3}+\frac{4}{n+2}-\frac{8}{n+3}$$

(ii) As $n\to\infty,\frac{4}{n+2}\to 0$ and $\frac{8}{n+3}\to 0$

$\therefore\ S_\infty=\frac{2}{3}$ (deduced)

(iiia)

$$\sum_{r=1}^{n}\frac{4r}{(r+2)(r+3)(r+4)}$$

(replace r by $r-1$)

$$=\sum_{r=2}^{n+1}\frac{4r-4}{(r+1)(r+2)(r+3)}$$

$$=\sum_{r=1}^{n+1}\frac{4r-4}{(r+1)(r+2)(r+3)}-\frac{4(1)-4}{(1+1)(1+2)(1+3)}$$

$$=\frac{2}{3}+\frac{4}{n+3}-\frac{8}{n+4}$$

(iiib) For $r>1$,

$$r^2+4r+4>r^2+4r+3$$

$$(r+2)^3>(r+1)(r+2)(r+3)$$

$$\sum_{r=11}^{n}\frac{r-1}{(r+2)^3}<\frac{1}{4}\sum_{r=11}^{n}\frac{4r-4}{(r+1)(r+2)(r+3)}$$

$$\sum_{r=11}^{n}\frac{r-1}{(r+2)^3}<\frac{1}{4}\left[\sum_{r=1}^{n}\frac{4r-4}{(r+1)(r+2)(r+3)}-\sum_{r=1}^{10}\frac{4r-4}{(r+1)(r+2)(r+3)}\right]$$

$$\sum_{r=11}^{n}\frac{r-1}{(r+2)^3}<\frac{1}{4}\left[\left(\frac{2}{3}+\frac{4}{n+2}-\frac{8}{n+3}\right)-\left(\frac{2}{3}+\frac{4}{12}-\frac{8}{13}\right)\right]$$

$$\sum_{r=11}^{n}\frac{r-1}{(r+2)^3}<\frac{1}{4}\left[\frac{11}{39}+\frac{4}{n+2}-\frac{8}{n+3}\right]$$

$$\sum_{r=11}^{n}\frac{r-1}{(r+2)^3}<\frac{11}{156}-\frac{n+1}{(n+2)(n+3)}$$

$$\sum_{r=11}^{n}\frac{r-1}{(r+2)^3}<\frac{11}{156}\quad\left(\because\frac{n+1}{(n+2)(n+3)}>0 \text{ as } n>0\right)$$

12 Sketch on a single Argand diagram the loci of $|z+3+4\text{i}|=5$ and $|z+3+4\text{i}|=|z+6|$. [3]

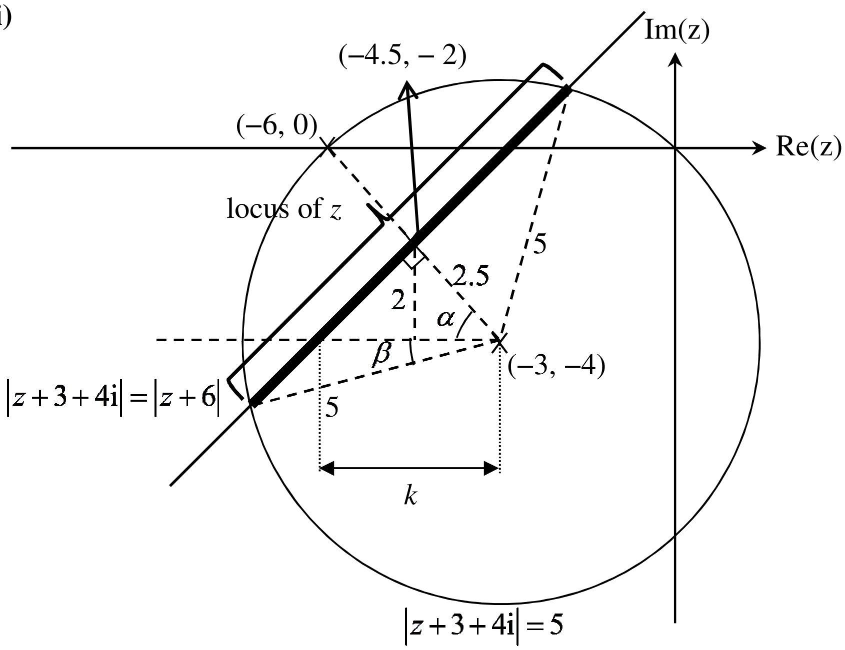
(i) Hence indicate clearly on the Argand diagram the locus of z that satisfies the relation $|z+3+4\text{i}|\le 5$ and $|z+3+4\text{i}|=|z+6|$. [1]

(ii) Given that $-\pi<\arg(z+3+4\text{i})\le\pi$, find the exact range of values of $\arg(z+3+4\text{i})$. [4]

(iii) Find the exact value of z where $|\arg(z+3+4\text{i})|$ is as large as possible. [2]

Solution

(i)



(ii) $\cos(\alpha+\beta)=\frac{2.5}{5}=\frac{1}{2}$

So $\alpha+\beta=\frac{\pi}{3}$

And $\sin\alpha=\frac{2}{2.5}=\frac{4}{5}$

$\beta=\frac{\pi}{3}-\sin^{-1}\frac{4}{5}$

So range of $-\pi<\arg(z+3+4\text{i})\le-\pi+\beta$ or $\pi-\left(\frac{\pi}{3}+\alpha\right)\le\arg(z+3+4\text{i})\le\pi$

$$-\pi < \arg(z+3+4\mathrm{i}) \le -\frac{2\pi}{3} - \sin^{-1}\left(\frac{4}{5}\right) \text{ or } \frac{2\pi}{3} - \sin^{-1}\frac{4}{5} \le \arg(z+3+4\mathrm{i}) \le \pi$$

(iii) $\cos\alpha = \dfrac{3}{5} = \dfrac{2.5}{k}$

$\cos\alpha = \dfrac{3}{5} = \dfrac{2.5}{k}$

$k = \dfrac{25}{6}$

Value of z is $\left(-\dfrac{43}{6}\right) - 4\mathrm{i}$

13 A curve is defined by the equation $y^2 = \ln(\mathrm{e} - x)$, $y > 0$ and $x < \mathrm{e}$.

(i) Show that $2y\dfrac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^{-y^2} = 0$. [2]

(ii) By further differentiation, show that $\left(2+4y^2\right)\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$. [2]

(iii) Hence, find the Maclaurin's series of y, up to and including the term in x^3, giving your answers in exact form. [3]

(iv) Find the Maclaurin's series of the curve defined by the equation

$$[y(2-x)]^2 = \ln(1-x) + 1.$$ [3]

Solution

(i) Differentiating implicitly,

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{\mathrm{e}-x}$$

From $y^2 = \ln(\mathrm{e} - x)$,

$$\Rightarrow \mathrm{e}^{y^2} = \mathrm{e} - x$$

$$\Rightarrow \frac{-1}{\mathrm{e}-x} = -\mathrm{e}^{-y^2}$$

$\therefore 2y\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-1}{\mathrm{e}-x}$

$\Rightarrow 2y\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\mathrm{e}^{-y^2} \Rightarrow 2y\dfrac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^{-y^2} = 0$

(ii) Differentiating implicitly,

$$\left(2\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2y\mathrm{e}^{-y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

Since $\mathrm{e}^{-y^2} = -2y\dfrac{\mathrm{d}y}{\mathrm{d}x}$, the equation becomes

$$2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y^2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 0$$

Factorizing,

$$\left(2+4y^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$$

(iii) Differentiating implicitly,

$$(8y)\left(\frac{dy}{dx}\right)^3+(2+4y^2)2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+2y\frac{d^3y}{dx^3}=0$$

When $x = 0$,

$y^2 = \ln e = 1 \Rightarrow y = 1$ (since $y > 0$).

$$2\frac{dy}{dx}=\frac{-1}{e}\Rightarrow\frac{dy}{dx}=-\frac{1}{2}e^{-1}$$

$$(2+4)\left(-\frac{1}{2}e^{-1}\right)^2+2\frac{d^2y}{dx^2}=0\Rightarrow\frac{d^2y}{dx^2}=-\frac{3}{4}e^{-2}$$

$$8\left(-\frac{1}{2}e^{-1}\right)^3+14\left(-\frac{1}{2}e^{-1}\right)\left(-\frac{3}{4}e^{-2}\right)+2\frac{d^3y}{dx^3}=0\Rightarrow\frac{d^3y}{dx^3}=-\frac{17}{8}e^{-3}$$

Hence, the Maclaurin's series of the curve is

$$1-\frac{1}{2}e^{-1}x-\frac{3}{4}e^{-2}\frac{x^2}{2!}-\frac{17}{8}e^{-3}\frac{x^3}{3!}+\ldots = 1-\frac{1}{2e}x-\frac{3}{8e^2}x^2-\frac{17}{48e^3}x^3+\ldots$$

(iv) By replacing x with ex in the original equation,

$$y^2 = \ln(e - ex) = \ln(e(1-x)) = \ln(1-x)+1$$

Hence, replacing x with ex in the Maclaurin's series,

$$y(2-x) = 1-\frac{1}{2}x-\frac{3}{8}x^2-\frac{17}{48}x^3+\ldots$$

$$\Rightarrow y=(2-x)^{-1}\left(1-\frac{1}{2}x-\frac{3}{8}x^2-\frac{17}{48}x^3+\ldots\right)$$

$$\Rightarrow y=\frac{1}{2}(1-\frac{x}{2})^{-1}\left(1-\frac{1}{2}x-\frac{3}{8}x^2-\frac{17}{48}x^3+\ldots\right)$$

$$\Rightarrow y=\frac{1}{2}\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\ldots\right)\left(1-\frac{1}{2}x-\frac{3}{8}x^2-\frac{17}{48}x^3+\ldots\right)$$

$$\Rightarrow y=\frac{1}{2}\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}-\frac{1}{2}x-\frac{1}{4}x^2-\frac{1}{8}x^3-\frac{3}{8}x^2-\frac{3}{16}x^3-\frac{17}{48}x^3+\ldots\right)$$

$$\Rightarrow y=\frac{1}{2}-\frac{3}{16}x^2-\frac{13}{48}x^3+\ldots$$

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