1 A sequence of negative numbers $u_1, u_2, u_3, \ldots$ is defined by $u_1 = -2$ and $u_{n+1} = \dfrac{8}{u_n - 4}$, $n \geq 1$.

(i) Given that the sequence converges to l, determine the exact value of l. **[2]**

(ii) Use a calculator to determine the behaviour of the sequence. **[2]**

2 **(a)** Describe fully a sequence of transformations which would transform the curve $y = \dfrac{1}{x}$ onto the curve $y = \dfrac{2x+a}{x-a}$, where a is a positive constant. **[3]**

(b) Sketch the curve $y^2 = \dfrac{2x+a}{x-a}$, where a is a positive constant, stating the equations of any asymptotes and the coordinates of any points where the curve crosses the x- and y-axes. **[2]**

3 **(i)** Given that $\mathrm{f}(x) = 2 + \ln(1+\sin x)$, find $\mathrm{f}(0)$, $\mathrm{f}'(0)$ and $\mathrm{f}''(0)$. Hence write down the Maclaurin series for $\mathrm{f}(x)$, up to and including the term in x^2. **[3]**

(ii) Use the first three terms of the Maclaurin series for $\mathrm{f}(x)$ to find the series expansion for $\mathrm{g}(x)$, where $\mathrm{g}(x) = \left[2 + \ln(1+\sin x)\right]^{\frac{1}{2}}$, up to and including the term in x^2. **[2]**

(iii) Use your answer in part **(ii)** to give an approximation for $\int_0^{\frac{\pi}{2}} \mathrm{g}(x)\,\mathrm{d}x$. Find, also, an accurate value for $\int_0^{\frac{\pi}{2}} \mathrm{g}(x)\,\mathrm{d}x$, to three significant figures. **[2]**

4 Show that $k^2 + k - 1 = (k+1)(k+2) - 2(k+2) + 1$. **[1]**

Hence, by using the method of differences, show that

$$\sum_{k=1}^{n} \frac{k^2+k-1}{(k+2)!} = \frac{1}{2} - \frac{n+1}{(n+2)!}.$$ **[3]**

Hence find $\displaystyle\sum_{k=4}^{n+3} \frac{(k-2)^2 + k - 3}{k!}$. **[3]**

5 The function f is defined by

$$\mathrm{f} : x \mapsto \frac{a}{|x|+1}, \quad x \in \mathbb{R}, \quad x \geq -1,$$

where a is a positive constant.

(i) Show that f^{-1} does not exist. **[2]**

(ii) If the domain of f is restricted to $x \geq k$, where $k \in \mathbb{R}$, state the least value of k such that f has an inverse. Define f^{-1} in a similar form. **[3]**

(iii) Hence, or otherwise, find the value(s) of x, in terms of a, such that the composite function f^2 satisfies the equation $\mathrm{f}^2(x) = x$. **[3]**

6 The equation of a curve C is $y = 2x + \dfrac{8k^2}{x-2}$, where $k > 0$.

(i) Prove, using an algebraic method, that y cannot lie between two values (to be determined in terms of k). **[3]**

(ii) Sketch C for the case where $k = 1$, stating the equations of any asymptotes and the coordinates of any turning points. **[3]**

(iii) State, with a reason, the range of values of b, where $b > 0$, such that the graph of $b^2x^2 - 4y^2 = 4b^2$ does not intersect C. **[2]**

7 **(a)** A function f is defined by

$$\mathrm{f}(x) = \begin{cases} (x-2)^2 - 4 & \text{for} \quad 0 \leq x < 5 \\ -\dfrac{5}{3}x + \dfrac{40}{3} & \text{for} \quad 5 \leq x < 8 \end{cases}$$

and $\mathrm{f}(x) = \mathrm{f}(x+8)$ for all real values of x.

(i) Sketch the graph of $y = \mathrm{f}(x)$ for $-3 \leq x \leq 16$. **[2]**

(ii) Find the exact value of the area bounded by the curve and the x-axis between $x = -3$ and $x = 13$. **[3]**

(b) The region bounded by the curve $y^2 = \dfrac{\sqrt{x-3}}{x+13}$ and the lines $x = 3$ and $x = 19$ is rotated through π radians about the x-axis. Using the substitution $t = \sqrt{x-3}$, find the volume of the solid obtained in exact form. **[4]**

8 During winter, it is known that the temperature inside a building decreases at a rate proportional to the positive difference between the temperature inside the building and the temperature outside the building.

(i) Express this information as a differential equation relating T and t, where T is the temperature inside the building at time t, for the case where the temperature outside has the constant value T_0. **[1]**

(ii) Show that $T = T_0 + Ae^{-kt}$, where A and k are positive constants. **[2]**

Initially the temperature inside the building was set at $25°C$. However, the heater in the building broke down and it was only discovered at 4.00 pm, when the temperature inside the building had already dropped to $18°C$. By 4.30 pm, the temperature inside the building had dropped to $12°C$. The temperature outside the building is assumed to be constant at $-2°C$.

(iii) Show that $A = 27$. Sketch a solution curve for $t > 0$. **[3]**

(iv) Find, to the nearest minute, the time at which the heater broke down. **[3]**

9 **(a)** The plane Π_1 has equation $x + y + z = 3$. The points A and B have coordinates $(1,\ 2,\ 3)$ and $(2,\ -3,\ -2)$ respectively.

(i) Find the perpendicular distance from A to Π_1. **[2]**

The line l passes through the point A. The line l is also parallel to $\overrightarrow{OB}$.

(ii) Find the acute angle between l and Π_1. **[2]**

Let Q be the point of intersection of l and Π_1.

(iii) Without finding the coordinates of Q, deduce the length of AQ correct to 3 significant figures. **[1]**

(b) Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, and let C be a point on AB such that $AC : BC = \lambda : 1 - \lambda$.

(i) Express $\overrightarrow{OC}$ in terms of λ, $\mathbf{a}$ and $\mathbf{b}$. **[1]**

It is known that $|\mathbf{a}| = 2|\mathbf{b}|$ and $\angle AOB = \theta$, where $0 \le \theta \le \dfrac{\pi}{2}$.

(ii) Given that $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$, find $\cos\theta$ in terms of λ. **[4]**

(iii) Hence, find the exact value of λ when ΔACO and ΔAOB are similar triangles. **[2]**

10 It is given that $z_1 = 1 + c\mathrm{i}$, where c is a positive real number.

(i) Find $z_1^{\,3}$, in terms of c. **[1]**

(ii) Given that $1 + c\mathrm{i}$ is a root of the equation

$$z^3 - 7z^2 + kz - 15 = 0,$$

find the values of the real numbers c and k. **[3]**

(iii) For these values of c and k, solve the equation in part **(ii)**. **[1]**

The complex number z_2 is given by $z_2 = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$.

(iv) On a single Argand diagram, sketch the loci

(a) $\left| z - z_2 \right| = 2$ **[1]**

(b) $\arg\left(z - z_2^{\,*} \right) = \dfrac{\pi}{3}$ **[2]**

(v) Hence find the complex number that satisfies both $\left| z - z_2 \right| = 2$ and $\arg\left(z - z_2^{\,*} \right) = \dfrac{\pi}{3}$, in the form $a + \mathrm{i}b$, where a and b are real exact values. **[2]**

11 The equations of three planes Π_1, Π_2 and Π_3 are given as follows.

$$\begin{aligned} \Pi_1 &: \quad x + y + z = 3 \\ \Pi_2 &: \quad x + ky + 2z = 4 \\ \Pi_3 &: \quad x + \beta y + 3z = \mu \end{aligned}$$

The point $(p,\ 0,\ q)$ lies on both Π_1 and Π_2.

(i) Find the values of p and q. **[2]**

The planes Π_1 and Π_2 meet at line l.

(ii) By using your answer in **(i)** or otherwise, find the equation of line l in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{m},\quad \lambda \in \mathbb{R}$, leaving your answer in terms of k. **[2]**

Hence, what can be said about the values for β and μ (in terms of k when necessary) when

(iii) the three planes intersect each other at a point, **[1]**

(iv) the three planes intersect each other in a line. **[2]**

Explain your answers clearly for parts **(iii)** and **(iv)**.

It is now known that $k = \beta = \mu - 4$.

(v) Explain why the three planes will always have at least one common point. **[2]**

12 **(a)** A petrol tanker is damaged in a road accident, and petrol leaks onto a flat section of a motorway. The leaking petrol begins to spread in a circle of thickness 2 mm. Petrol is leaking from the tanker at a rate of 0.0084 m^3s^{-1}. Find the rate at which the radius of the circle of petrol is increasing at the instant when the radius of the circle is 3 m, giving your answer in m s^{-1} to 2 decimal places. **[4]**

(b)

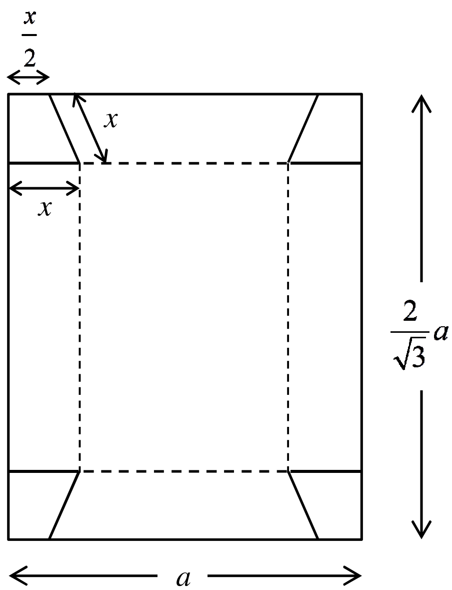

Fig. 1

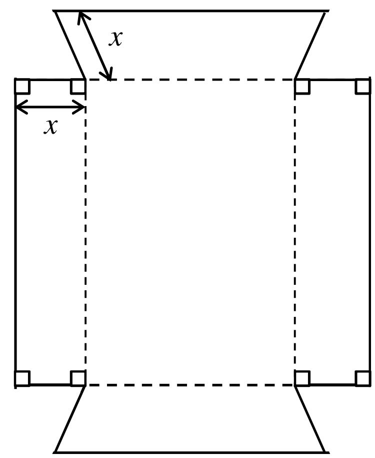

Fig. 2

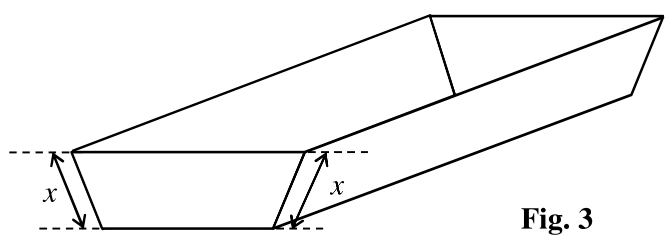

Fig. 3

Fig. 1 above shows a piece of card of dimension a by $\frac{2}{\sqrt{3}}a$. A trapezium is cut from each corner to give the shape shown in Fig. 2. The remaining card is folded along the dotted lines, to form the shape shown in Fig. 3.

(i) Show that the volume of the shape in Fig. 3 is $V = x\left(a - \frac{3}{2}x\right)^2$. **[3]**

(ii) Use differentiation to find, in terms of a, the maximum value of V, proving that it is a maximum. **[5]**

End of Paper