SRJC 2014 H2 Math Preliminary Exam Paper 2

Section A: Pure Mathematics [40 marks]

1 The line ℓ with Cartesian equation $\dfrac{4-x}{4}=\dfrac{z}{3}$, $y = 1$ contains the point B with position vector $\mathbf{j} + 3\mathbf{k}$. A point A, not lying on ℓ, has position vector $2\mathbf{i} + (1+\sqrt{5})\mathbf{j} - \mathbf{k}$.

(i) Given that $\mathbf{c}$ denotes a unit vector parallel to ℓ, find $\left|\overrightarrow{AB}\cdot\mathbf{c}\right|$ and give a geometrical interpretation of this quantity. [3]

(ii) Hence find the shortest distance from A to ℓ. [2]

The foot of perpendicular from A to ℓ is denoted by F and the foot of perpendicular from F to AB is denoted by G.

(iii) Write down the ratio between the area of ΔAGF and area of ΔBGF. [1]

(iv) Hence, deduce the ratio AG:GB and find the position vector of G. [2]

Solution

(i) $\dfrac{4-x}{4}=\dfrac{z}{3}$, $y = 1 \Rightarrow \dfrac{x-4}{-4}=\dfrac{z-0}{3}$, $y = 1$.

Vector equation of ℓ is $\mathbf{r}=\begin{pmatrix}4\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}-4\\0\\3\end{pmatrix}$, $\lambda\in\mathbb{R}$.

A unit vector $\mathbf{c}$ parallel to ℓ = $\dfrac{1}{5}\begin{pmatrix}-4\\0\\3\end{pmatrix}$ (OR $\dfrac{1}{5}\begin{pmatrix}4\\0\\-3\end{pmatrix}$)

$$\left|\overrightarrow{AB}\bullet\mathbf{c}\right|=\left|\left(\begin{pmatrix}0\\1\\3\end{pmatrix}-\begin{pmatrix}2\\1+\sqrt{5}\\-1\end{pmatrix}\right)\bullet\frac{1}{5}\begin{pmatrix}-4\\0\\3\end{pmatrix}\right|$$

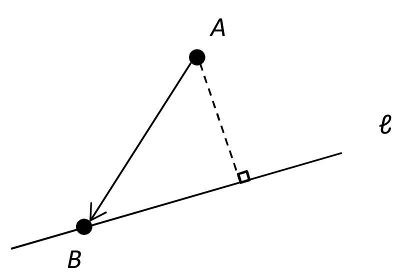
$$=\frac{1}{5}\left|\begin{pmatrix}-2\\-\sqrt{5}\\4\end{pmatrix}\bullet\begin{pmatrix}-4\\0\\3\end{pmatrix}\right|$$

$=4$

$\left|\overrightarrow{AB}\bullet\mathbf{c}\right|$ is the length of projection of $\overrightarrow{AB}$ onto ℓ (or onto $\mathbf{c}$).

(iii)

$$\left|\overrightarrow{AB}\right|=\left|\begin{pmatrix}-2\\-\sqrt{5}\\4\end{pmatrix}\right|=5$$



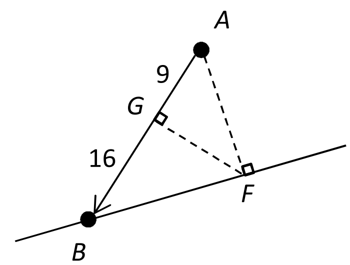
By Pythagoras' theorem,

The shortest distance from A to $\ell = \sqrt{5^2 - 4^2} = 3$

(iv) AGF and ΔBGF are similar triangles.

Since AF corresponds to BF,

$$\frac{\text{Area of }\Delta\text{AGF}}{\text{Area of }\Delta\text{BGF}} = \frac{3^2}{4^2} = \frac{9}{16}$$



(v) $\dfrac{\text{Area of }\Delta AGF}{\text{Area of }\Delta BGF} = \dfrac{\frac{1}{2}(AG)(GF)}{\frac{1}{2}(BG)(GF)} = \dfrac{AG}{BG}$

From above, $\dfrac{AG}{BG} = \dfrac{9}{16}$

$$\overrightarrow{OG} = \frac{9\overrightarrow{OB} + 16\overrightarrow{OA}}{25} = \frac{1}{25}\left(9\begin{pmatrix}0\\1\\3\end{pmatrix} + 16\begin{pmatrix}2\\1+\sqrt{5}\\-1\end{pmatrix}\right)$$

$$= \frac{1}{25}\begin{pmatrix}32\\25+16\sqrt{5}\\11\end{pmatrix}$$

2 **(a)** **(i)** By using a graphic calculator, find the x-coordinates of the points of intersection of the curves $y = e^x$ and $y = 2x + 1$.

Hence solve the inequality $e^x < 2x + 1$. [2]

(ii) Hence, find the exact value of $\int_{-2}^{1} \left| e^x - 2x - 1 \right| \mathrm{d}x$. [3]

(b) Find $\int \dfrac{2x+1}{x^2 - 4x + 7}\,\mathrm{d}x$. [3]

(c) Find $\int \sin x \ln(\cos x)\,\mathrm{d}x.$. [2]

Solution

(ai) $x = 0$ or $x = 1.26$

$0 < x < 1.26$

$e^x - 2x - 1 < 0$ for $0 < x < 1$ and $e^x - 2x - 1 > 0$ for $-2 < x < 0$

(ii) $\int_{-2}^{1} \left| e^x - 2x - 1 \right| \mathrm{d}x = \int_{-2}^{0} \left(e^x - 2x - 1\right)\mathrm{d}x - \int_{0}^{1} \left(e^x - 2x - 1\right)\mathrm{d}x$

$=\left[e^x-x^2-x\right]_{-2}^{0}-\left[e^x-x^2-x\right]_{0}^{1}$

$=6-e^{-2}-e$

(b) $\int \frac{2x+1}{x^2-4x+7}\,\mathrm{d}x = \int \frac{2x-4+5}{x^2-4x+7}\,\mathrm{d}x$

$= \int \frac{2x-4}{x^2-4x+7}\,\mathrm{d}x + \int \frac{5}{(x-2)^2+3}\,\mathrm{d}x$

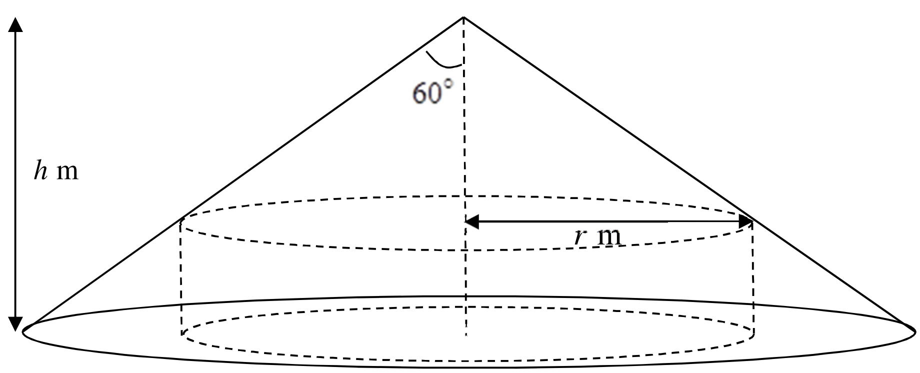
$=\ln\left(x^2-4x+7\right)+\frac{5}{\sqrt{3}}\tan^{-1}\left(\frac{x-2}{\sqrt{3}}\right)+c$

(c) $\int \sin x \ln(\cos x)\ \mathrm{d}x = -\cos x \ln(\cos x) - \int (-\cos x)\left(\frac{-\sin x}{\cos x}\right)\ \mathrm{d}x$

$=-\cos x \ln(\cos x) - \int \sin x\ \mathrm{d}x$

$=-\cos x \ln(\cos x) + \cos x + C$

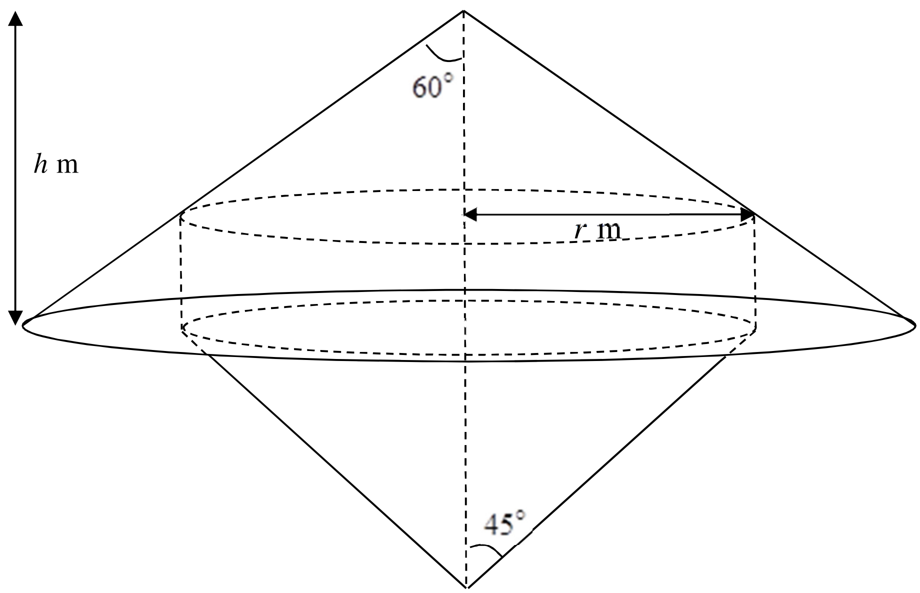
3



A right circular cone-shaped structure of fixed height h m and semi-vertical angle 60° stands on horizontal ground. A cylindrical tank of radius r m, which is fully filled with a type of liquid chemical, is inscribed inside the cone.

(i) Find the value of r in terms of h when the cylindrical tank has a maximum volume. [5]

An engineer decides to build the above structure with cylindrical tank of radius r as found in (i). To prevent the liquid chemical from contaminating the ground when leakages occur, an inverted cone of semi-vertical angle of 45° will be attached to the cylindrical tank as shown in the diagram below.



If a crack is found at the centre of the base of the cylindrical tank and the liquid chemical is leaking into the inverted cone at a rate of 0.3 m^3/min,

(ii) find the exact rate of change of surface area of the liquid chemical in the inverted cone half an hour after the leaking starts. [5]

Solution

(i)

Let d be the vertical length from the vertex of the cone to the top of the cylindrical tank.

$V = \pi r^2 (h - d)$

Since $\tan 60^\circ = \frac{r}{d}$, we have $d = \frac{r}{\sqrt{3}} = \frac{\sqrt{3}}{3} r$.

Hence, $V = \pi r^2 \left(h - \frac{\sqrt{3}}{3} r \right)$

$V = \pi r^2 h - \frac{\sqrt{3}}{3} \pi r^3$

$\frac{\mathrm{d}V}{\mathrm{d}r} = 2\pi r h - \sqrt{3}\pi r^2$

$2\pi r h - \sqrt{3}\pi r^2 = 0$

$\pi r \left(2h - \sqrt{3} r \right) = 0$

$\therefore r = 0$ (rejected since $r > 0$) or $r = \frac{2}{\sqrt{3}} h = \frac{2\sqrt{3}}{3} h$

$\frac{\mathrm{d}^2V}{\mathrm{d}r^2} = 2\pi h - 2\sqrt{3}\pi r$

When $r = \frac{2\sqrt{3}}{3} h$,

$$\frac{\mathrm{d}^2V}{\mathrm{d}r^2} = 2\pi h - 2\sqrt{3}\pi\left(\frac{2\sqrt{3}}{3}h\right)$$
$$= 2\pi h - 4\pi h$$
$$= -2\pi h \text{ (< 0 since } h \text{ is positive)}$$

Hence, the cylindrical tank has a maximum volume when $r = \frac{2\sqrt{3}}{3}h$.

(ii) Let x m be the radius of the water surface in the inverted cone at time t mins and L be the volume of the water in the inverted cone at time t mins. Since the angle is 45°, when the radius of the water surface is x m, the height of the water in the inverted cone is x m as well.

$A = \pi x^2 \quad \Rightarrow \quad \frac{\mathrm{d}A}{\mathrm{d}x} = 2\pi x$

$L = \frac{1}{3}\pi x^3 \quad \Rightarrow \quad \frac{\mathrm{d}L}{\mathrm{d}x} = \pi x^2$

30 mins after leaking, volume of chemical in the container $= 0.3 \times 30 = 9\,\mathrm{m}^3$

$$9 = \frac{1}{3}\pi(x)^3$$
$$x = 3\pi^{-\frac{1}{3}}$$
$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}L}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$$
$$0.3 = \pi\left(3\pi^{-\frac{1}{3}}\right)^2 \frac{\mathrm{d}x}{\mathrm{d}t}$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\pi^{-\frac{1}{3}}}{30}\,\mathrm{m/min}$$
$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$$
$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi\left(3\pi^{-\frac{1}{3}}\right)\left(\frac{\pi^{-\frac{1}{3}}}{30}\right)$$
$$= \frac{1}{5}\pi^{\frac{1}{3}}\ \mathrm{m}^2/\mathrm{min}$$

Therefore, the area of liquid surface is increasing at a rate of $\frac{1}{5}\pi^{\frac{1}{3}}\ \mathrm{m}^2/\mathrm{min}$.

4 **(a)** The complex number z is such that $\left|z^2\right| = 3$ and $\arg(-\mathrm{i}z) = \frac{\pi}{4}$.

Find w in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$, if $|wz| = 2\sqrt{3}$ and $\arg\left(\frac{z^2}{w}\right) = \frac{5}{6}\pi$. [4]

(b) Solve the equation $z^4 = -81$, giving the roots in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [2]

(i) Hence, express $z^4 + 81$ as the product of two quadratic factors with real [3] coefficients, giving each factor in exact non-trigonometrical form.

The roots of the equation $z^4 = -81$ are represented by z_1, z_2, z_3, z_4 such that $\arg(z_1) < \arg(z_2) < \arg(z_3) < \arg(z_4)$.

(ii) Explain why the locus of all points z such that $|z - z_3| = |z - z_2|$ passes through the origin. [1]

(iii) The points A, B, C and D represent the complex numbers z_1, z_2, v and z_4 respectively, with $v = -\frac{1}{2} z_3$.

Find the area enclosed by the points A, B, C and D. [2]

Solution

(a) $\left|z^2\right| = 3 \Rightarrow |z| = \sqrt{3}$

$$\arg(-iz) = \frac{\pi}{4} \Rightarrow \arg(-i) + \arg(z) = \frac{\pi}{4}$$

$$\arg(z) = \frac{\pi}{4} - \left(-\frac{\pi}{2}\right)$$

$$= \frac{3\pi}{4}$$

$$|wz| = 2\sqrt{3}$$

$$|w||z| = 2\sqrt{3}$$

$$\therefore |w| = 2$$

$$\arg\left(\frac{z^2}{w}\right) = \frac{5}{6}\pi$$

$$2\arg(z) - \arg(w) = \frac{5}{6}\pi$$

$$\arg(w) = 2\left(\frac{3}{4}\pi\right) - \frac{5}{6}\pi$$

$$= \frac{2}{3}\pi$$

$$w = 2\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$$

$$= -1 + \sqrt{3}i$$

(b) $z^4 = -81 = 81e^{i\pi} = 81e^{i(\pi + 2n\pi)}$

$$z = 81^{\frac{1}{4}} e^{i\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)}, \; n = 0, \pm 1, -2$$

$$= 3e^{i\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)}$$

(i) $z^4 + 81$

$$= (z - 3e^{\frac{\pi}{4}i})(z - 3e^{-\frac{\pi}{4}i})(z - 3e^{\frac{3\pi}{4}i})(z - 3e^{-\frac{3\pi}{4}i})$$

$= (z^2 - 6\,z\cos\left(\frac{\pi}{4}\right) + 3^2)\,(z^2 - 6\,z\cos\left(\frac{3\pi}{4}\right) + 3^2)$

$= (z^2 - 3\sqrt{2}\ z\ + 9)(z^2 + 3\sqrt{2}\,z + 9)$

(ii) Given $|z - z_3| = |z - z_2|$ ------------ (1)

For z=0+0i, we have $|0 - z_3| = |z_3| = 3$

For z=0+0i, we have $|0 - z_2| = |z_2| = 3 = |0 - z_3|$

$\therefore$ (0, 0) is one of the locus points for (1)

Hence, the locus $|z - z_3| = |z - z_2|$ passes through the origin.

(iii) w_3 is obtained from z_3 by rotating π radian about origin and a scaling by a scale factor of half in the direction of OZ_1.

So required area $= \frac{1}{2}\left[(6)(3) - \left(6 \times \frac{3}{2}\right)\right]$

$= \frac{9}{2}\,\text{units}^2$

Section B: Statistics [60 marks]

5 **(a)** Five couples are seated in a row. Find the number of ways in which three particular men must not be seated next to each other. [2]

(b) These five couples are now seated at a round table.

(i) Find the number of ways in which the wives must be seated next to her husbands. [1]

(ii) Find the number of ways in which the five women are not all seated together. [2]

(c) A funfair game consists of a segment which requires the player to draw coloured balls from a box. The box contains five blue balls and two red balls. In order for the player to win the game, he has to pick two red balls consecutively. Whenever a blue ball is drawn, it will be replaced back into the box and the drawing continues. However, when a red ball is drawn, it will not be replaced back into the box. Calculate the probability that the player wins the game. [3]

Solution

(a)

Number of arrangements = $(7!)\times(8C3)\times 3! = 1693440$ ways

Alternatively,

Number of arrangements = No. of arrangements without restrictions −
No. of arrangements with all three men seated together −
No. of arrangements with two of the three men seated together

$= 10! - (8!\times 3!) - (7!\times 8C2\times 2!\times 3C2\times 2!)$

$= 1693440$ ways

(b)(i)

Number of arrangements = $(4!)\times(2!)^5 = 768$ ways

(b)(ii)

Number of arrangements = $(9!) - (5!\times 5!) = 348480$ ways

(c)

$$\text{Total probability} = \left(\frac{2}{7}\right)\left(\frac{1}{6}\right) + \left(\frac{5}{7}\right)\left(\frac{2}{7}\right)\left(\frac{1}{6}\right) + \left(\frac{5}{7}\right)^2\left(\frac{2}{7}\right)\left(\frac{1}{6}\right) + \ldots$$

$$= \left(\frac{2}{7}\right)\left(\frac{1}{6}\right)\left[1 + \left(\frac{5}{7}\right) + \left(\frac{5}{7}\right)^2 + \ldots\right]$$

$$= \left(\frac{1}{21}\right)\left[\frac{1}{1-\left(\frac{5}{7}\right)}\right]$$

$$= \frac{1}{6}$$

6 In a high school, 94% of its students own a Friend Book account. A random sample of 80 students is taken from the high school. The random variable X denotes the number of students in the sample who own a Friend Book account.

(i) State, in the context of this question, an assumption needed to model this situation by a binomial distribution. [1]

(ii) If the sample has at least 70 students who own a Friend Book account, find the probability that there are at most 74 students who own a Friend Book account in the sample. [2]

(iii) Estimate the probability that there are exactly 75 students who own a Friend Book account in the sample. [3]

(iv) Sixty samples each of size 80 is taken from the high school. Find the probability that the average number of students who own a Friend Book account in each sample exceeds 76. [2]

Solution

(i) Assumption: The probability of a student owning an account is assumed to be constant at 0.94 or the event that students owning a Friend Book account are independent of one another.

(ii) $X \sim B(80, 0.94)$

$P(X \le 74 \mid X \ge 70)$

$= \dfrac{P(70 \le X \le 74)}{P(X \ge 70)}$

$= \dfrac{P(X \le 74) - P(X \le 69)}{1 - P(X \le 69)}$

= 0.342

(iii) The random variable Y denotes the number of students in the sample who **do not** own a Friend Book account. $Y \sim B(80, 0.06)$

Since n is large ($n > 50$) and $np = 4.8$ (< 5), $X \sim Po(4.8)$ approximately.

P(exactly 75 students who own an account)

= P(exactly 5 students who do not own an account)

= P(Y = 5)

= 0.175

(iv) $\mathrm{E}(X) = 80 \times 0.94 = 75.2$

$\mathrm{Var}(X) = 80 \times 0.94 \times 0.06 = 4.512$

Since n is large, by CLT, $\bar{X} \sim N\left(75.2, \dfrac{4.512}{60}\right)$ approximately.

Using G.C., $P(\bar{X} > 76) \approx 0.00177$

7 **(a)** A famous zoologist Elsa claims that the mean tail length of Proboscis Monkeys is at most 65 cm on a particular remote island. The tails of a random sample of 20 Proboscis Monkeys are measured and found to have mean 65.5 cm and standard deviation 0.9 cm. Test at the 1% significance level whether Elsa's claim is valid. [5]

(b) Another famous zoologist Anna claims that the mean tail length of Proboscis Monkeys on another island is 63 cm. The tails of a new random sample of 25 Proboscis Monkeys on the island have been measured and the mean length of these monkeys is found to be t cm.

(i) Assuming that the tail lengths of Proboscis Monkeys on the island are normally distributed with standard deviation 5.8 cm, find the set of values of t for Anna's claim to be rejected at the 5% level of significance. [4]

(ii) Explain 5% level of significance in the context of the question. [1]

Solution

(a) $s^2 = \frac{20}{19}(0.9^2) = 0.85263$

Let μ be the mean tail length of Proboscis Monkeys on the island

To test H_o : $\mu = 65$ (Elsa's claim)

Against H_1: $\mu > 65$

Using right tailed test at 1% level of significance

Since the sample size $n = 20$ is small, population variance is unknown and assuming the tail lengths of Proboscis Monkeys on the island follows a normal distribution,

Under Ho, $T = \dfrac{\bar{X} - 65}{\sqrt{\dfrac{s^2}{20}}} \sim t(19)$

Using GC, p – value = 0.0128.

Since p – value > 0.01, we do not reject Ho and there is insufficient evidence to conclude that the mean tail lengths of Proboscis Monkeys on the island is greater than 65 cm, thus Elsa's claim is valid at 1% level of significance.

(bi) Let μ be the mean tail length of Proboscis Monkeys on the island

To test H_o : $\mu = 63$ (Anna's claim)

Against H_1: $\mu \neq 63$

Using 2 tailed test at 5% level of significance.

Under Ho, $\bar{X} \sim N(63, \frac{5.8^2}{25})$

Since Anna's claim (Ho) is rejected, $P\left(\bar{X} < t\right) < 0.025$ or $P\left(\bar{X} > t\right) < 0.025$

$P\left(\bar{X} < t\right) < 0.025$ or $P\left(\bar{X} < t\right) > 0.975$

$$P\left(Z < \frac{t-63}{\sqrt{\frac{5.8^2}{25}}}\right) < 0.025 \quad \text{or} \quad P\left(Z < \frac{t-63}{\sqrt{\frac{5.8^2}{25}}}\right) > 0.975$$

$$\frac{t-63}{\sqrt{\frac{5.8^2}{25}}} \leq -1.95996 \quad \text{or} \quad \frac{t-63}{\sqrt{\frac{5.8^2}{25}}} \geq 1.95996$$

$$t \leq 63 - 1.95996\sqrt{\frac{5.8^2}{25}} \quad \text{or} \quad t \geq 63 + 1.95996\sqrt{\frac{5.8^2}{25}}$$

$t \le 60.726$ or $t \ge 65.274$

The solution set = $\{t \in \mathbb{R} : t \le 60.7 \text{ or } t \ge 65.3\}$

(ii) It means that there is a 0.05 probability that the test will conclude that the mean tail length of the Proboscis Monkeys is not 63 cm when in fact the mean tail length of the monkeys is 63 cm.

OR

It means that there is a 0.05 probability that the claim "the mean tail length of Proboscis Monkeys is 63 cm" is rejected wrongly.

8 **(a)** It is given that the regression line y on x for the following bivariate data is $y = 6 + 0.5x$

x	20	22	24	26	28	30	32	34
y	14	19	16	a	20	22	25	18

Find a. [2]

(b) The data below show the average heights of Japanese maple trees planted in a botanical garden which were observed over a period of 10 years. It is assumed that the height of a tree is dependent on its age..

Age of trees in years (x)	1	2	3	4	5	6	7	8	9	10
Average Height in feet (y)	4	6.1	7.3	8.5	9.3	9.7	9.8	10.1	10.5	10.6

(i) Give a sketch of the scatter diagram for the above data for the model $y = a + bx$. [2]

(ii) For the model used in (i), give an interpretation, in the context, of the value of b. [1]

(iii) State, with reason, which of the following models among A, B or C is more appropriate for the given data.

A: $y = a + bx$ B: $y = a + b\sqrt{x}$ C: $y = a + b\ln x$

Write down the equation of the least-squares regression line for the chosen model, stating clearly the values of a and b. [3]

(iv) Using the model you have chosen from part (iii), calculate an estimate of the age, correct to two decimal places, of a Japanese maple tree whose height is 10 feet. [3]

[Turn Over

Comment on the reliability of the estimate.

Solution

(a) $\bar{x} = \dfrac{1+2+3+...+10}{8}$ and $\bar{y} = \dfrac{14+19+16+a+20+22+25+18}{8}$

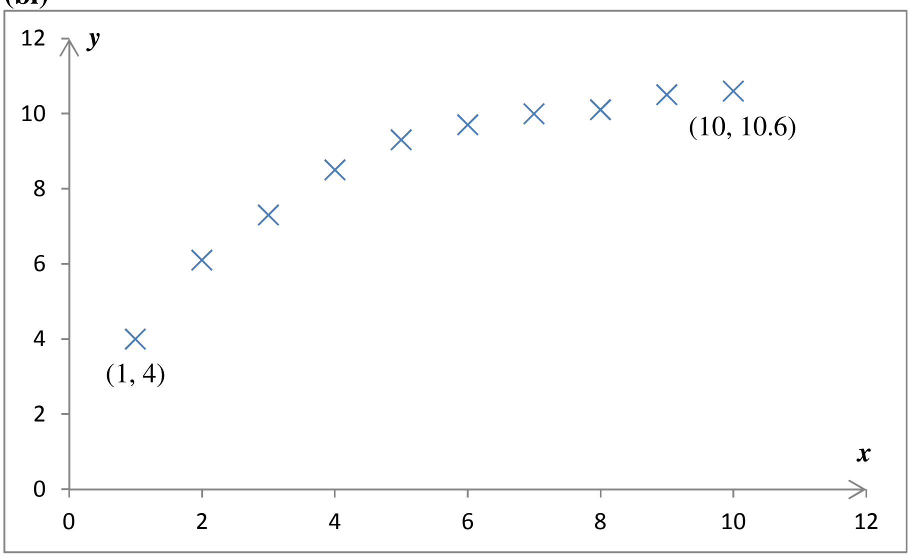
$= 27$ $\qquad = \dfrac{134+a}{8}$.

Since $(\bar{x}, \bar{y})$ lies on the regression line,

$\dfrac{134+a}{8} = 6 + 0.5(27)$

$a = 22$

(bi)



(ii) The value of b in the context of this question, is the rate of change of the predicted average height of the maple trees in the 10 year period, i.e. for every increase of 1 year in the age of the trees, the average height of the trees is expected to increase by b feet.

(iii) For A, $r = 0.920$, For B, $r = 0.967$ and for C, $r = 0.994$

The curvilinear shape of scatter diagram suggest that Model A (linear) is not suitable.

Since $|r|$ is the greatest for C, the most appropriate model is Model C.

Equation required is $y = 4.16 + 2.93 \ln x$

$a = 4.16$ and $b = 2.93$

(iv) Substitute $y = 10$ in the above equation,

$$10 = 4.1601 + 2.9329 (\ln x)$$

$x = 7.32$ years
Since $y = 10$ is within the range of values [4, 10.6], the estimate is an interpolation and is reliable as the values of y and $\ln x$ are strongly correlated ($r = 0.994$) there.

9 A busy bus stop serves 13 bus services, each having an average time arrival of one bus every 6 minutes. The number of buses from each bus service arriving at the bus stop during a fixed time interval is modelled by a Poisson distribution.

(i) State, in the context, one assumption for number of buses from each bus service arriving at the bus stop during a fixed time interval to be well modelled by a Poisson Distribution. [1]

(ii) Find the probability that at least 2 buses arrive at the bus stop in 1 minute. [2]

(iii) Find the most probable number of buses, k, arriving at the bus stop in a randomly chosen 1 minute period. [2]

(iv) In 22 randomly chosen 1-minute periods, find the probability that there are at least 11 such periods with k or more buses arriving at the bus stop. [2]

(v) Find the number of seconds, n, such that the probability that no bus arrives at the bus stop in n second is $e^{-1.3}$. [2]

(vi) Using a suitable approximation, find the probability that more than 380 buses arriving at the bus stop in 3 hours. [2]

Solution

(i) Buses arrive at the bus stop independently of every other buses.

OR

The average number of bus arriving at the bus stop is proportional to the time interval for each bus service serving the bus stop.

OR

No two buses may arrive at the bus stop simultaneously.

(ii) Let X be the random variable for the number of buses arriving at the bus stop in 1 minute.

Avg. no. of buses arriving in 1 min. $= 13 \times \frac{1}{6} = \frac{13}{6}$

$$X \sim \text{Po}\left(\frac{13}{6}\right)$$

$P(X \geq 2) = 1 - P(X \leq 1) = 0.6372 = 0.637$ (3 s.f.)

(iii) From GC,

$P(X = 1) = 0.2482$

$P(X = 2) = 0.2689$

$P(X = 3) = 0.1942$

Since when X equals to 2 it gives the highest probability, the most likely number of buses arriving in a 1 minute interval is 2. Therefore $k = 2$.

(iv) P(2 or more buses waiting at the bus bay) $= P(X \geq 2) = 0.6372$

Let Y be the random variable for the number of days with 3 or more buses waiting simultaneously at the bus bay, out of 22 one minute periods.

$$Y \sim \mathrm{B}(22, 0.6372)$$

$\mathrm{P}(Y \geq 11) = 1 - \mathrm{P}(Y \leq 10) = 0.9383 = 0.938$ (3 s.f.)

(v) Let W be the random variable for the number of buses arriving at the bus stop in n seconds.

$$W \sim \mathrm{Po}\left(\frac{13}{360}n\right)$$

$\mathrm{P}(W = 0) = \mathrm{e}^{-1.3}$

$$\mathrm{e}^{-\frac{13n}{360}} \frac{\left(\frac{13n}{360}\right)^0}{0!} = \mathrm{e}^{-1.3} \Rightarrow \mathrm{e}^{-\frac{13n}{360}} = \mathrm{e}^{-1.3}$$

$$\Rightarrow n = 36$$

(vi) Let V be the random variable for the number of buses arriving at the bus stop in 3 hours.

$$V \sim \mathrm{Po}(390)$$

Since $\lambda = 390 > 10$ is large, $V \sim \mathrm{N}(390, 390)$ approximately.

$\mathrm{P}(V > 380) \xrightarrow{c.c.} \mathrm{P}(V > 380.5) = 0.6847 = 0.685$ (3 s.f.)

10 **(a)** The random variables X and Y have the distributions $\mathrm{N}(a, 25)$ and $\mathrm{N}(75, b)$ respectively, where $a, b \in \mathbb{R}$. It is given that Y is related to X by the formula $Y = cX + d$, where $c \in \mathbb{R}^+$, $d \in \mathbb{R}$ and $7\mathrm{P}(75 \leq Y < 77) = \mathrm{P}(Y > 73)$. Find b, c and obtain an equation involving a and d. **[6]**

(b) Fragrance rice is sold in two types of packaging, namely standard and large. The mass of each standard packet of fragrance rice is a normal random variable with mean 5 kg and standard deviation 30 g. The mass of each large packet of fragrance rice is a normal random variable with mean 10 kg and standard deviation 50 g.

(i) Find the probability that the average mass of 1 large packet and 4 standard packets of fragrance rice is between 1.01 kg and 1.02 kg more than a standard packet of fragrance rice. **[4]**

(ii) Six standard packets of fragrance rice are randomly chosen. Find the probability that the sixth packet is the fourth packet with mass less than 4.99 kg. **[2]**

Solution

(a) $\mathrm{P}(Y < 73) = \dfrac{5}{12}$

$$\mathrm{P}\left(Z < \frac{-2}{\sqrt{b}}\right) = \frac{5}{12}$$

$$\frac{-2}{\sqrt{b}} = -0.210428$$

$b = 90.334$

$b = 90.3(3\text{ s.f})$

$\mathrm{Var}(Y) = c^2 \mathrm{Var}(X)$

$90.334 = 25c^2$

$c = 1.9009 (\because c > 0)$

$c = 1.90 (\because c > 0)$

$\mathrm{E}(Y) = c\mathrm{E}(X) + d$

$75 = 1.90a + d$

(bi)

Let S denote the mass of a randomly chosen standard packet of fragrance rice.

Let L denote the mass of a randomly chosen large packet of fragrance rice.

$S \sim \mathrm{N}\left(5000, 30^2\right)$, $L \sim \mathrm{N}\left(10000, 50^2\right)$

$\dfrac{L + S_1 + S_2 + S_3 + S_4}{5} - S \sim \mathrm{N}(1000, 1144)$

$\mathrm{P}\left(1010 < \dfrac{L + S_1 + S_2 + S_3 + S_4}{5} - S < 1020\right)$

$= 0.107$

(ii) $\dfrac{5!}{2!3!}\left[\mathrm{P}(S < 4990)\right]^4 \left[\mathrm{P}(S \geq 4990)\right]^2$

$= 0.0741$

END OF PAPER