2014 H2 Maths Prelim Paper 2 Solutions

1	$\dfrac{x}{x-2}-\dfrac{4}{(x-2)^2}\le 0,\ x\ne 2$ $\Rightarrow \dfrac{x^2-2x-4}{(x-2)^2}\le 0 \Rightarrow \dfrac{(x-1)^2-5}{(x-2)^2}\le 0$ $\Rightarrow (x-1)^2-5\le 0,$ since $(x-2)^2$ is always positive for all real values of *x*. $\Rightarrow (x-1)^2\le 5$ $\Rightarrow \lvert x-1\rvert \le \sqrt{5}$ $\Rightarrow -\sqrt{5}\le x-1\le \sqrt{5}$ $\Rightarrow 1-\sqrt{5}\le x\le 1+\sqrt{5},\ \ x\ne 2$ $\therefore 1-\sqrt{5}\le x<2$ or $2<x\le 1+\sqrt{5}$ (I) For $\dfrac{e^x}{e^x+2}\le \dfrac{4}{(e^x+2)^2}$, Replace *x* by $-e^x$ in (I), $1-\sqrt{5}\le -e^x<2$ or $2<-e^x\le 1+\sqrt{5}$ (inadmissible since $-e^x<0$ for all $x\in\mathbb{R}$) $\therefore 1-\sqrt{5}\le -e^x$ & $-e^x<2$ $\Rightarrow e^x\le \sqrt{5}-1$ & $e^x>-2$ $\Rightarrow x\le \ln(\sqrt{5}-1)$ & $x\in\mathbb{R}$ $\Rightarrow x\le \ln(\sqrt{5}-1)$	
2	Let $\sin x+\sin 3x+\sin 5x+\cdots+\sin(2n-1)x=\sum_{r=1}^{n}\sin(2r-1)x$ Let P(*n*) be the proposition that $\sum_{r=1}^{n}\sin(2r-1)x=\dfrac{1-\cos(2nx)}{2\sin x}$ for all $n\in\mathbb{Z}^+$. When $n=1$, LHS = $\sin x$ RHS = $\dfrac{1-\cos(2x)}{2\sin x}=\dfrac{2\sin^2 x}{2\sin x}=\sin x$ = LHS (double angle formula for sin) $\therefore \mathrm{P}(1)$ is true. Assume that P(*k*) is true for some $k\in\mathbb{Z}^+$ i.e. $\sum_{r=1}^{k}\sin(2r-1)x=\dfrac{1-\cos(2kx)}{2\sin x}$	

	To prove that P(k+1) is true i.e. $\sum_{r=1}^{k+1}\sin(2r-1)x=\dfrac{1-\cos(2(k+1))x}{2\sin x}$ $\text{LHS}=\sum_{r=1}^{k+1}\sin(2r-1)x$ $=\sum_{r=1}^{k}\sin(2r-1)x+\sin(2(k+1)-1)x$ $=\dfrac{1-\cos(2kx)}{2\sin x}+\sin(2k+1)x$ $=\dfrac{1-\cos(2kx)+2\sin x\sin(2k+1)x}{2\sin x}$ $=\dfrac{1-\cos(2kx)+[-\cos(2k+2)x+\cos(2kx)]}{2\sin x}$ $=\dfrac{1-\cos(2k+2)x}{2\sin x}$ $=\dfrac{1-\cos(2(k+1))x}{2\sin x}$ $=\text{RHS}$ $\text{P}(k)$ is true $\Rightarrow$ $\text{P}(k+1)$ is true Since P(1) is true, and $\text{P}(k)$ is true $\Rightarrow$ $\text{P}(k+1)$ is true , by Mathematical Induction, P(n) is true for all $n\in\mathbb{Z}^+$.	
3 (a)(i)	$\frac{n}{2}[2(5)+(n-1)(1)]=\frac{m}{2}[2(7)+(m-1)(2)]$ $n[9+n]=m[12+2m]$ $n^2+9n=12m+2m^2$ $n^2+9n+(-2m^2-12m)=0$ $n=\dfrac{-9\pm\sqrt{9^2-4(1)(-2m^2-12m)}}{2(1)}$ $=\dfrac{-9\pm\sqrt{81+8m^2+48m}}{2}$ Since n must be positive, $n=\dfrac{-9+\sqrt{81+8m^2+48m}}{2}$ (shown)	
3(a) (ii)	Using the GC,	

[key in $Y_1 = \dfrac{-9+\sqrt{81+8X^2+48X}}{2}$ and using 2nd table,

$X = 15$, $Y_1 = 21$]

$n = 21$, $m = 15$.

3(b)(i)

After the 1st payment, the amount owed = $(1500 - x)(1.04)$

If the whole amount owed is paid off after 2nd payment x,

We have

$x = (1500 - x)(1.04)$

$\therefore\ x = 764.71$

3(b)(ii)

Repayment schedule	Repayment 15th of each month	Amount still owed after repayment, with interest charged
1st	x	$1.04(1500 - x)$
2nd	x	$1.04(1.04(1500\text{-}x)\text{-}x) =$ $1.04^2(1500) - 1.04x(1+1.04)$
3rd	x	$1.04^3(1500) - 1.04x(1+1.04+1.04^2)$
…	x	…
(n)th	x	$1.04^{(n)}\ (1500)$ $- 1.04x(1+1.04+\ldots+1.04^{n-1})$

If the whole amount owed is paid off fully after the (n+1)th payment,

That means the (n+1)th repayment, x, equals to the amount still owed with increase after nth repayment:

$x = 1.04^{(n)}\ (1500) - 1.04x(1+1.04+\ldots+ 1.04^{n-1})$ --- (*)

$x(1+1.04+1.04^2+\ldots+1.04^n) = 1.04^{(n)}1500$

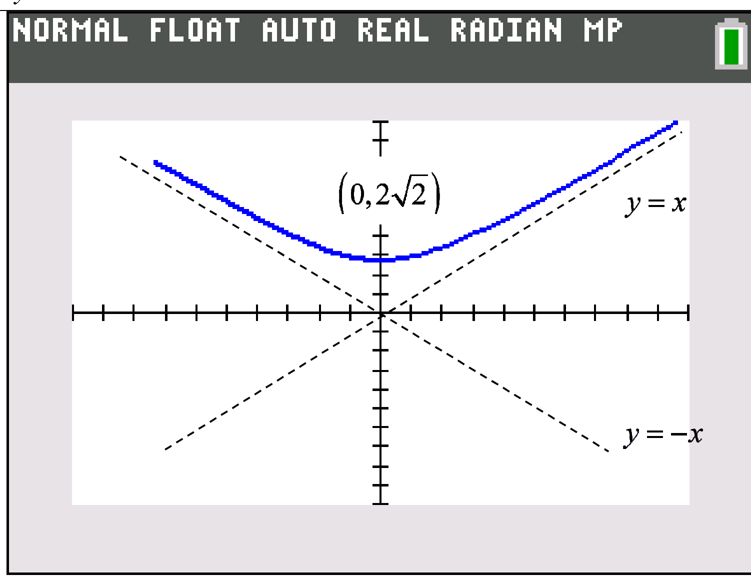
$x\left(\dfrac{((1.04)^{n+1}-1)}{1.04-1}\right) = 1.04^n(1500)$

$x = \dfrac{1500(1.04)^n(1.04-1)}{(1.04)^{n+1}-1}$

$= \dfrac{60(1.04)^n}{(1.04)^{n+1}-1}$

$\therefore A = 60$, $r = 1.04$. (Ans)

4(a)	$\dfrac{\mathrm{i}z}{z-2z^{*}-2}=-1$ $\mathrm{i}z=-z+2z^{*}+2$ Let $z = x + y\mathrm{i}$ $\mathrm{i}(x+y\mathrm{i})=-(x+y\mathrm{i})+2(x-y\mathrm{i})+2$ $-y+x\mathrm{i}=(x+2)-3y\mathrm{i}$ Equating real & imaginary parts, $y=-(x+2)$ ----------- (1) $x=-3y$ ----------- (2) Solving (1) & (2), $x = -3$, $y = 1$ Hence, $z = -3 + \mathrm{i}$	
4(b)(i)	$\dfrac{z}{z-r}=\dfrac{r\mathrm{e}^{\mathrm{i}\theta}}{r\mathrm{e}^{\mathrm{i}\theta}-r}$ $=\dfrac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta}-1}$ $=\dfrac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}(\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}-\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}})}$ $=\dfrac{\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}}{2\mathrm{i}\sin(\frac{\theta}{2})}$ $=\dfrac{\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}}{2\mathrm{i}\sin\frac{\theta}{2}}$ (Note: $\frac{1}{\mathrm{i}}=-\mathrm{i}$) $=\frac{1}{2}-\frac{1}{2}\mathrm{i}\cot(\frac{\theta}{2})$.	
4(b)(ii)	$z^{3}=-27$ $=27\mathrm{e}^{\mathrm{i}(\pi)}$ $z=3\,\mathrm{e}^{\mathrm{i}(\frac{\pi+2k\pi}{3})}$, $k = -1, 0, 1$ Therefore the roots are $z=3\mathrm{e}^{\mathrm{i}(-\frac{\pi}{3})}, 3\mathrm{e}^{\mathrm{i}(\frac{\pi}{3})}, 3\mathrm{e}^{\mathrm{i}(\pi)}$.	
4(b)(iii)	$(\dfrac{3w}{w-1})^{3}=-27$ Using results from (i), let $z=\dfrac{3w}{w-1}$ z (w-1) = 3w	

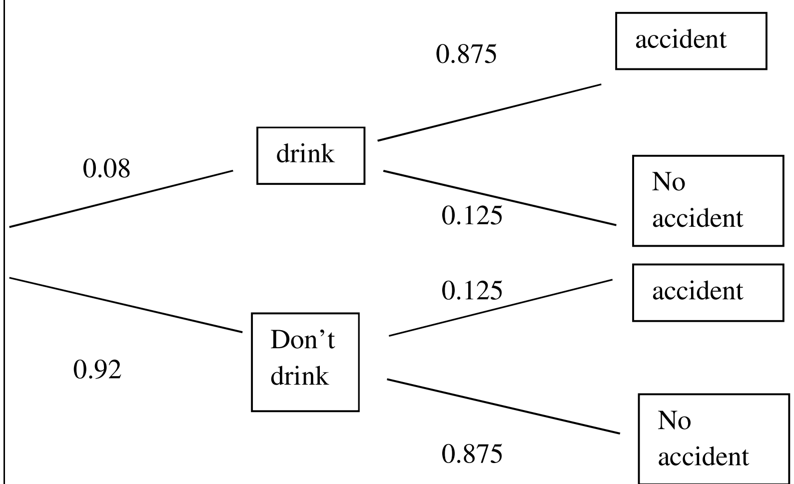
	$w(z-3)=z$ $w=\dfrac{z}{z-3}$ Since $\lvert z\rvert=r=3$, Therefore, $w=\dfrac{1}{2}-\dfrac{1}{2}\mathrm{i}\cot(\dfrac{\theta}{2})$, where $\theta=-\dfrac{\pi}{3},\dfrac{\pi}{3},\pi$ Substituting, $w=\dfrac{1}{2}+\mathrm{i}(\dfrac{1}{2}\sqrt{3}),\dfrac{1}{2}-\mathrm{i}(\dfrac{1}{2}\sqrt{3}),\dfrac{1}{2}$.	
5(i)	$x^2=\left(2t-\dfrac{1}{t}\right)^2=4t^2+\dfrac{1}{t^2}-4$ $y^2=\left(2t+\dfrac{1}{t}\right)^2=4t^2+\dfrac{1}{t^2}+4$ $y^2-x^2=8$	
5(ii)	 Since $y^2=x^2+8$ As $x\to\pm\infty$, $y^2\to x^2$ $\therefore y\to\pm x$ $x=0$ $2t-\dfrac{1}{t}=0$ $t=\dfrac{1}{\sqrt{2}}$ $y=2t+\dfrac{1}{t}=\dfrac{2}{\sqrt{2}}+\sqrt{2}=2\sqrt{2}$	

	$\frac{dy}{dx}=0$ $2t^2-1=0$ $t=\frac{1}{\sqrt{2}}$ Min point = y intercept = $\left(0,2\sqrt{2}\right)$	
5(iii)	$\frac{dx}{dt}=2+\frac{1}{t^2}$; $\frac{dy}{dt}=2-\frac{1}{t^2}$ $\frac{dy}{dx}=\frac{2-\frac{1}{t^2}}{2+\frac{1}{t^2}}=\frac{2t^2-1}{2t^2+1}$	
5(iv)	Equation of tangent at P: $y-\left(2p+\frac{1}{p}\right)=\frac{2p^2-1}{2p^2+1}\left(x-\left(2p-\frac{1}{p}\right)\right)$ substitute $x=0, y=1$ $1-\left(2p+\frac{1}{p}\right)=\frac{2p^2-1}{2p^2+1}\left(0-\left(2p-\frac{1}{p}\right)\right)$ $-1+2p+\frac{1}{p}=\frac{2p^2-1}{2p^2+1}\left(\frac{2p^2-1}{p}\right)$ $\left(-p+2p^2+1\right)\left(2p^2+1\right)=\left(2p^2-1\right)^2$ $-2p^3-p+4p^4+2p^2+2p^2+1=4p^4-4p^2+1$ $2p^3-8p^2+p=0$ $p(2p^2-8p+1)=0$ $p=0$(reject as p>0), $2p^2-8p+1=0$ $p=\frac{8\pm\sqrt{56}}{4}=2+\frac{\sqrt{14}}{2}$ or $2-\frac{\sqrt{14}}{2}$ (reject since the point P is in the first quadrant) x-coordinate of the point P = $2\left(2+\frac{\sqrt{14}}{2}\right)-\frac{1}{2+\frac{\sqrt{14}}{2}}=4+\sqrt{14}-\frac{2}{4+\sqrt{14}}=2\sqrt{14}$ Required area $=\int_0^{2\sqrt{14}} y\ dx=\int_{\frac{1}{\sqrt{2}}}^{2+\frac{\sqrt{14}}{2}}\left(2t+\frac{1}{t}\right)\frac{dx}{dt}dt=\int_{\frac{1}{\sqrt{2}}}^{2+\frac{\sqrt{14}}{2}}\left(2t+\frac{1}{t}\right)\left(2+\frac{1}{t^2}\right)dt$ $=36.7$ units^2 (correct to 3 s.f.)	

6	$P(X \le 100+a) = 0.8$ $P\left(Z \le \frac{100+a-\mu}{\sigma}\right) = 0.8$ $\frac{100+a-\mu}{\sigma} = 0.84162 \cdots (1)$ $P(X \le 100-a) = 0.3$ $P\left(Z \le \frac{100-a-\mu}{\sigma}\right) = 0.3$ $\frac{100-a-\mu}{\sigma} = -0.52440 \cdots (2)$ $(1)-(2)$, $\frac{2a}{\sigma} = 1.3660$ $\sigma = 1.4641a \cdots (3)$ Sub (3) into (1), $100+a-\mu = 0.84162(1.4641a)$ $\mu = 100 - 0.232a$ $k = 0.232$	
7(i)	A simple random sample may not be appropriate as the sample obtained may not be representative of the loans in the loan portfolio and consist of a very high proportion of loans from one of the categories, e.g. Personal Loans. This may not reflect the soundness of the portfolio appropriately.	
7(ii)	The following method is to be used for stratified sampling: 1. The number of loans that are to be taken from each category of loans is: General Commerce: $0.25 \times 500 = 125$; Building Construction: $0.6 \times 500 = 300$; Personal Loans: 75 2. Use simple random sampling within each stratum to obtain the required loans by numbering the loans in the stratum, say Personal Loans, and using a random number generator to obtain the sample of 75 loans. Repeat this for the other strata.	
8	Let A be the event a motorist drinks before driving and B be the event that a motorist got into an accident.	

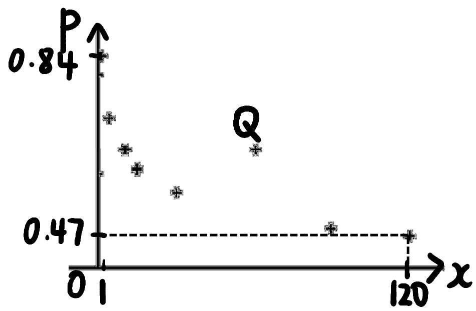
Table in 7(ii):

General Commerce	Building Construction	Personal Loans
$0.25 \times 500 = 125$	$0.6 \times 500 = 300$	75

	$P(A)=0.08$ $P(A')=0.92$ $P(A'\cap B)=0.115$ $P(B\mid A')=\dfrac{P(A'\cap B)}{P(A')}=\dfrac{0.115}{0.92}=0.125$	
8(i)	 $P(B)=0.875\times0.08+0.125\times0.92$ $=0.185$ <u>Alternative Solution</u>: $P(B\mid A)=0.125\times7=0.875$ $P(B)=P(B\mid A)P(A)+P(B\mid A')P(A')$ $=0.875\times0.08+0.125\times0.92$ $=0.185$	
8(ii)	Required Probability $=P(A\mid B)=\dfrac{P(A\cap B)}{P(B)}$ $=\dfrac{0.08\times0.875}{0.185}=0.378$ (3 s.f.)	
9(i)	Total number of ways without restriction $=9!=362880$ Total number of ways $=7!\times3!$ Required Probability $=\dfrac{7!\times3!}{9!}=0.0833$ (3 s.f.)	
9(ii)	Required Probability = 1- P(no girls beside man) <u>Case 1</u>: man at the ends	

	M B _ _ _ _ _ _ _ _ or _ _ _ _ _ _ _ _ B M Number of ways = 2cases $\times {}^5C_1 \times 7! = 10 \times 7!$ Case 2: man not at ends, seated between 2 boys _ $\boxed{\text{B M B}}$ _ _ _ _ _ Number of ways $= {}^5C_2 \times 2! \times 7! = 20 \times 7!$ Required Probability $= 1 - \dfrac{(10+20)\times 7!}{9!} = 1 - 0.41667 = 0.583$ (3 s.f.)	
9(iii)	Required Prob =1- P(no boy on positions left of man) Case 1: Man on left most. M _ _ _ _ _ _ _ _ _ Number of ways =8! Case 2: man on seat 2 and 1 girl on left: G M _ _ _ _ _ _ _ _ Number of ways = 3x7! Case3: man on seat 3 and 2 girls on left: G G M _ _ _ _ _ _ _ Number of ways = ${}^3C_2 \times 2 \times 6!$ Case 4: Man on seat 4 and 3 girls on left: G G G M _ _ _ _ _ _ Number of ways= $3! \times 5!$ Required Prob $= 1 - \dfrac{8! + 3\times 7! + 6\times 6! + 3! \times 5!}{9!} = 1 - 0.166666 = 0.833$ (3 s.f.)	
10(i)	$\bar{x} = \dfrac{\sum x}{n} = \dfrac{110.5}{60} = 1.8417 = 1.84$ (3 s.f.)	

	$s^2=\frac{1}{n-1}\left[\sum x^2-\frac{\left[\sum x\right]^2}{n}\right]$ $=\frac{1}{59}\left[220.3-\frac{12210.25}{60}\right]$ $=0.28468$ $=0.285$ (3 s.f.)	
10(ii)	Let X be the random variable "time-interval between trains during peak hours in minutes" and μ be the population mean. Test $\mathrm{H}_0:\mu=a$ $\mathrm{H}_1:\mu>a$ Under H_0, $\overline{X}\sim N(a,\frac{0.28468}{60})$, by Central Limit Theorem approximately We use a right-tailed z-test at 5% level of significance. When H_0 is rejected, $P(\overline{X}>\overline{x})\le 0.05$ $\Rightarrow P\left(Z>\frac{1.8417-a}{\sqrt{\frac{0.28468}{60}}}\right)\le 0.05$ $\frac{1.8417-a}{\sqrt{\frac{0.28468}{60}}}\ge 1.64485$ $a\le 1.73$ Hence for the author's claim to be valid, $0<a\le 1.73$.	
10(iii)	Test $\mathrm{H}_0:\ \mu=1.4$ $\mathrm{H}_1:\ \mu\ne 1.4$ Assuming that X is normally distributed Under H_0, $T=\frac{\bar{X}-\mu}{S/\sqrt{n}}\sim t(6)$ Use **a two-tailed t-test at** 5% level of significance. p-value $=0.087619>0.05$ Since $p>5\%$, we **do not reject** H_0. There is **insufficient** evidence at the 5% significance level to say that the mean time interval between trains during peak hours differs from 1.4 minutes.	

11(i)	 There is an outlier at (60, 0.65), thus it is most likely to be incorrect.	
11(ii)	For $p = a + \frac{b}{x}$, the r-value between p and $\frac{1}{x}$ is 0.85558. For $p = a + b\ln x$, the r-value between p and ln x is -0.99787. Since the magnitude of the r-value for $p = a + b\ln x$ is closer to 1 than that for $p = a + \frac{b}{x}$, $p = a + b\ln x$ is a better model. From GC, $p = -0.077400\ln x + 0.83207$. a = 0.832 (3 s.f.) and b = - 0.0774 (3 s.f.)	
11(iii)	Since x is the independent variable we use the linear regression line p on $\ln x$ $p = -0.0774\ln x + 0.83207$ Since $p = 0.48$, $0.48 = -0.0774\ln x + 0.83207$ $0.0774\ln x = 0.83207 - 0.48$ $x = 94.51$ (to 2 decimal places) As the value of p = 0.48 is within the data range, the corresponding estimated value of x is within the data range. The r-value is close to 1. Hence, the estimate given by the regression line is reliable.	
12(i)	The probability of an egg being damaged is constant. Whether an egg is damaged or not is independent of that of any other egg.	
12(ii)	$X \sim \mathrm{B}\left(30, \frac{p}{100}\right)$	

	$P(X \le 1) = 0.87945$ $P(X = 0) + P(X = 1) = 0.87945$ $\left(1 - \frac{p}{100}\right)^{30} + 30\left(\frac{p}{100}\right)\left(1 - \frac{p}{100}\right)^{29} = 0.87945$ $(100 - p)^{30} + 30(p)(100 - p)^{29} = 0.87945 \times 100^{30}$ $(100 - p)^{29}(100 + 29p) = 8.7945 \times 10^{59}$ Using GC, $p = 2.00$	
12(iii)	$P(X > 1) = 1 - P(X \le 1) = 1 - 0.87945 = 0.12055$ Let Y be the number of trays with more than 1 damaged egg out of 40 trays $Y \sim \mathrm{B}(40, 0.12055)$ Since n is large, $np = 40(0.12055) = 4.8218 < 5$, $Y \sim \mathrm{P_o}(4.8218)$ approximately $P(Y \ge 4) = 1 - P(Y < 4)$ $= 1 - P(Y \le 3)$ $= 0.70909$ $= 0.709$	
12(iv)	Let A be the number of rejected cartons out of 100 cartons. $A \sim \mathrm{B}(100, 0.709)$ $\mathrm{E}(A) = 100(0.709) = 70.9$ $\mathrm{Var}(A) = 100(0.709)(1 - 0.709) = 20.6319$ Since n is large, by Central Limit Theorem, $\overline{A} = \frac{A_1 + \cdots + A_{52}}{52} \sim \mathrm{N}\left(70.9, \frac{20.6319}{52}\right)$ approximately $\overline{A} \sim \mathrm{N}(70.9, 0.39677)$ $\mathrm{P}(\overline{A} \le 71) = 0.563$	
13(i)	Average rate of occurrence of calls is constant over the time period. Calls to the customer service representatives occur independently.	
13(ii)	Let X and Y be the number of calls Mary and Jane receive in a 30-minute period respectively. $X \sim \mathrm{P_o}(1)$ $Y \sim \mathrm{P_o}(2)$ $T = X + Y \sim \mathrm{P_o}(3)$	

	Required probability $=\left[\mathrm{P}\left(T<5\right)\right]^4$ $=\left[\mathrm{P}\left(T\le 4\right)\right]^4$ $=\left(0.81526\right)^4$ $=0.442$	
13(iii)	$\mathrm{P}\left(X\ge 3 \mid T<5\right)$ $=\dfrac{\mathrm{P}\left(X\ge 3 \text{ and } T<5\right)}{\mathrm{P}\left(T<5\right)}$ $=\dfrac{\mathrm{P}\left(X=3 \text{ and } Y=0\right)+\mathrm{P}\left(X=3 \text{ and } Y=1\right)+\mathrm{P}\left(X=4 \text{ and } Y=0\right)}{\mathrm{P}\left(T<5\right)}$ $=\dfrac{\mathrm{P}\left(X=3\right)\mathrm{P}\left(Y=0\right)+\mathrm{P}\left(X=3\right)\mathrm{P}\left(Y=1\right)+\mathrm{P}\left(X=4\right)\mathrm{P}\left(Y=0\right)}{\mathrm{P}\left(T<5\right)}$ $=0.0331$	
13(iv)	Let C and D be the total number of calls Mary and Jane receive in 40 work days respectively. $C\sim \mathrm{P_o}(2(8)(40))$ $C\sim \mathrm{P_o}(640)$ Since $\lambda=640>10$, $C\sim \mathrm{N}\left(640,640\right)$ approximately $D\sim \mathrm{P_o}\left((4)(8)(40)\right)$ $D\sim \mathrm{P_o}\left(1280\right)$ Since $\lambda=1280>10$, $D\sim \mathrm{N}\left(1280,1280\right)$ approximately $D-2C\sim \mathrm{N}\left(0,3840\right)$ $\mathrm{P}\left(D\ge 2C\right)=\mathrm{P}\left(D-2C\ge 0\right)\xrightarrow{\text{c.c.}}\mathrm{P}\left(D-2C\ge -0.5\right)=0.503$	