Section A: Pure Mathematics [40 marks]

1 Solve the inequality

$$\frac{x}{x-2} \le \frac{4}{(x-2)^2}$$

giving your answer in the exact form. [3]

Hence solve $\dfrac{e^x}{e^x+2} \le \dfrac{4}{(e^x+2)^2}$. [2]

2 Use the method of mathematical induction to prove the following result:

$$\sin x + \sin 3x + \sin 5x + \cdots + \sin(2n-1)x = \frac{1-\cos(2nx)}{2\sin x},$$

where $0 < x < \dfrac{\pi}{2}$ and $n \in \mathbb{Z}^+$. [5]

3 **(a)** Benny and Aaron are making donuts. Benny takes 5 minutes to complete his first donut, and for each subsequent donut he makes, he takes 1 minute longer. Aaron takes 7 minutes to complete his first donut, and for each subsequent donut he makes, he takes 2 minutes longer.

If Benny and Aaron take the same amount of time to make n and m donuts respectively,

(i) Show that $n = \dfrac{-9+\sqrt{81+8m^2+48m}}{2}$. [3]

(ii) State the least values of n and m, where n and m are positive integers. [1]

(b) At the end of a particular month, Sandra owes the bank \$1500.
On the 15th day of the next month, she pays a fixed amount of \$$x$ to the bank where $x < 1500$. The bank charges interest at a rate of 4% to the remaining amount still owed at the end of the month. This process continues every month until the money owed is fully repaid to the bank.

(i) If the whole amount owed is paid off exactly after the second payment, show that the value of x is 764.71. [1]

(ii) If the whole amount owed is paid off exactly after the $(n+1)$th payment, show that the value of x is given by

$$x = \frac{Ar^n}{r^{n+1}-1},$$

where A and r are constants to be determined exactly. [4]

4 **(a)** Find the complex number z in the form $x + y\mathrm{i}$ where $x, y \in \mathbb{R}$ such that

$$\frac{\mathrm{i}z}{z-2z^{*}-2}=-1.$$ [3]

(b) **(i)** Show that for any complex number $z = r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$,

$$\frac{z}{z-r}=\frac{1}{2}-\frac{1}{2}\mathrm{i}\cot\left(\frac{\theta}{2}\right).$$ [2]

(ii) Find in the form of $r\mathrm{e}^{\mathrm{i}\theta}$, where r and θ are exact, the roots of the equation $z^3 = -27$. [3]

(iii) Using the results from **parts (i)** & **(ii)**, solve the equation $\left(\frac{3w}{w-1}\right)^3=-27$, expressing w in the form of $a + b\mathrm{i}$, where a and b are real numbers. [2]

5 A curve C has parametric equations

$$x=2t-\frac{1}{t},\qquad y=2t+\frac{1}{t}\qquad \text{where } t>0$$

(i) Find the Cartesian equation of the curve C. [2]

(ii) Sketch C, stating clearly the equations of any asymptotes. Show, on your diagram, the exact coordinates of any turning points and points of intersection with the axes. [2]

(iii) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. [2]

(iv) The tangent at the point P, with parameter p, in the first quadrant, intersects the y-axis at (0,1). Show that $p=2+\frac{\sqrt{14}}{2}$. Hence, find the area of the region bounded by the curve C, the x-axis, the y-axis and the vertical line passing through the point P. Give your answer correct to 3 significant figures. [5]

Section B: Statistics [60 marks]

6 The continuous random variable X has the distribution $\mathrm{N}\left(\mu,\sigma^2\right)$. It is known that $\mathrm{P}\left(X \le 100+a\right)=0.8$ and $\mathrm{P}\left(X \le 100-a\right)=0.3$. Express μ in the form of $100-ka$ where k is a constant to be determined. [4]

7 A bank has 50,000 loans in their loan portfolio. The loans are of the following categories: General Commerce, Building Construction and Personal Loans. The General Commerce loans consist of 25% of the total loans, and the Building Construction loans consist of 60% of the total loans. A risk manager would like to obtain a sample of 500 loans to check on the soundness of the bank's loan portfolio.

(i) Suggest a reason why simple random sampling would **not** be a preferred method to obtain the sample. [1]

(ii) Describe how you would be able to obtain a sample of loans using stratified sampling. [2]

8 A survey conducted by the traffic police found that 8% of motorists consumed alcohol before driving. It is known that the probability of a motorist who does not consume alcohol and gets into an accident is 0.115.

Show that the probability that the motorist gets into an accident given that he does not consume alcohol before driving is 0.125. [1]

Given that a motorist consumes alcohol before driving, he is 7 times more likely to get into an accident compared to one who does not consume alcohol before driving. By drawing a probability tree diagram, or otherwise, find the probability that

(i) a randomly chosen motorist got into an accident, [2]

(ii) a motorist consumed alcohol before driving, given that he got into an accident. [2]

9 A group of nine people consists of three girls, five boys and a man. The nine people are seated randomly in a row. Find the probability that

(i) the three girls are seated together. [2]

(ii) there is at least one girl seated beside the man. [2]

(iii) there is at least one boy seated to the left of the man. [3]

10 The Saint's Mass Rapid Transit (SMRT) claims that the mean time interval between consecutive train arrivals during peak hours is a minutes. A train auditor decides to challenge the claim and proceed to measure the time interval between train arrivals. A random sample of 60 time intervals (in minutes), was measured by the auditor. The results are summarised by $\sum x = 110.5$ and $\sum x^2 = 220.3$.

(i) Find the unbiased estimates of the population mean and variance. [2]

(ii) The auditor carries out a test at the 5% significance level and finds that the mean time interval between train arrivals has been underestimated by SMRT. State appropriate hypotheses for the test and find the range of values of a. [4]

(iii) It is given that $a = 1.4$. The auditor took another sample of 7 time intervals (in minutes) and the data are given as follows:

1.3 2.0 1.4 2.5 1.2 1.9 2.1

The auditor wishes to find out if the time interval differs from that claimed by SMRT using the new sample at the 5% level of significance. Perform the test, showing your workings clearly and state any assumptions that you have made. [4]

11 In a memory retention experiment, subjects were asked to memorise a list of items and then asked to recall the items at various times later. The proportion of items p correctly recalled at x minutes after the list was memorised was recorded in the following table.

x	1	5	10	15	30	60	90	120
p	0.84	0.71	0.65	0.61	0.56	0.65	0.49	0.47

(i) It was discovered that one of the results was incorrectly recorded. By drawing a scatter diagram, identify the result that is most likely to be incorrect and indicate the corresponding point on the scatter diagram by labelling it Q. [2]

It is thought that p can be modelled by one of the formulae

$$p = a + \frac{b}{x} \quad \text{or} \quad p = a + b\ln x$$

where a and b are constants.

The point Q is now removed for the following parts of this question.

(ii) By considering the values of the product moment correlation coefficient for both models, explain which of $p = a + \frac{b}{x}$ or $p = a + b\ln x$ is the better model for the remaining points in the given data set. Find the values of a and b for your chosen model. [3]

(iii) Use an appropriate regression line to estimate the value of x for $p = 0.48$, giving your answer correct to 2 decimal places. Comment on the reliability of your answer. [2]

12 In a large consignment of eggs, $p\%$ of the eggs is damaged.
Eggs are sold in trays of 30. The number of damaged eggs in a tray is denoted by X.

(i) State, in context, what must be assumed for X to be well modelled by a binomial distribution. [2]

Assume now that X has a binomial distribution.

(ii) Given that the probability of a tray containing at most 1 damaged egg is 0.87945, show that $(100-p)^{29}(100+29p) = 8.7945\times10^{59}$ and hence find p. [3]

(iii) A hawker bought 40 trays of eggs. A tray of eggs is rated Grade S if it contains more than 1 damaged egg. Using a suitable approximation, find the probability that the hawker bought at least 4 Grade S trays of eggs. [3]

(iv) Each week, a supermarket ordered 100 cartons, each consisting of 40 trays of eggs. A carton will be rejected if it contains at least 4 Grade S trays. Using the answer found in part **(iii)**, find the probability that the mean number of rejected cartons per week over a period of 52 weeks is not more than 71. [4]

13 Mary and Jane are two customer service representatives from a company.
Mary receives calls at an average rate of 2 calls per hour and Jane receives calls at an average rate of 4 calls per hour.

(i) State, in this context, two conditions that must be met for the number of calls received to be well modelled by a Poisson distribution. [2]

For the remaining parts of this question assume that these conditions are met. You should assume also that the customer service lines manned by Mary and Jane are independent of each other.

(ii) Find the probability that the total number of calls received by both customer service representatives is fewer than 5 calls in each of 4 successive 30-minute periods. [3]

(iii) Find the probability that Mary receives at least 3 calls in a randomly chosen 30-minute period, given that both customer service representatives receive a total of fewer than 5 calls. [3]

(iv) A typical work day consists of 8 hours of work. By using a suitable approximation, find the probability that Jane receives at least twice as many calls as Mary in a random sample of 40 work days. [4]

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