NANYANG JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION

Higher 2

MATHEMATICS **9740/01**

Paper 1 **16th September 2014**

3 Hours

Additional Materials: Cover Sheet
Answer Paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages.

1 The function f is defined by $\mathrm{f}: x \mapsto 2-(x+1)^2,\ x \in \mathbb{R},\ x \le k$.

(i) Determine the largest value of k for which the function f^{-1} exists. **[1]**

With this value of k,

(ii) find f^{-1} in a similar form. **[3]**

(iii) show algebraically that the x-coordinate of the point of intersection of the curves $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$ satisfies the equation $x^2+3x-1=0$ and find the value of this x-coordinate, correct to 3 decimal places. **[2]**

2 The region enclosed by the curve $y=\mathrm{e}^x \sin x$ where $0 \le x \le \frac{\pi}{2}$, the x-axis and the line $x=\frac{\pi}{2}$ is denoted by A. Find the exact area of A. **[4]**

Find the volume of revolution when the region bounded by the curves $y=\mathrm{e}^x \sin x$, $y=x+x^2+\frac{1}{3}x^3$ and the line $x=\frac{\pi}{2}$ is rotated completely about the x-axis. **[2]**

3 For a curve with equation $x^2+3xy+y^3=3$, find

(i) the coordinates of the point at which the tangent is parallel to the x-axis, **[4]**

(ii) the equation(s) of the normal(s) at $x=-1$. **[4]**

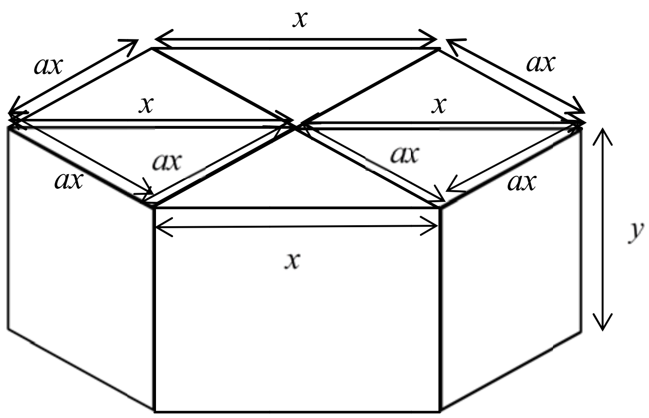
4 The complex number z satisfies the relation $\left|z-1+\sqrt{3}\mathrm{i}\right| \le 2$.

Illustrate, on an Argand diagram, the locus of points representing the complex number z. **[2]**

(i) Find the greatest possible value of $\arg\left(z+3\sqrt{3}\mathrm{i}\right)$. **[3]**

(ii) Find the range of values of θ, where $0 \le \theta \le \frac{\pi}{2}$, such that there exists a complex number w which satisfies the relations $\arg\left(w+3\sqrt{3}\mathrm{i}\right)=\theta$ and $\arg\left(w^*+2\right)=\theta$, where w^* is the conjugate of w. **[4]**

5 A company requires a box made of cardboard of negligible thickness to hold 300 cm^3 of powder when full. The top and the base of the box are made up of six identical isosceles triangles. The two identical sides of the isosceles triangle are of length ax cm, where a is a constant and $a > \frac{1}{2}$, and the remaining side is of length x cm. The height of the box is y cm (see diagram).



(i) Use differentiation to find, in terms of a, the value of x which gives a minimum surface area of the box. **[7]**

(ii) Show that, in this case, $\frac{y}{x} = 3\sqrt{\frac{2a-1}{2a+1}}$. Hence find the range of $\frac{y}{x}$. **[3]**

6 In a bid to analyse the path of an insect, an entomologist decides to fit a mathematical model for the path of the insect. The insect's path was observed for 20 seconds. The path travelled by the insect measured with respect to the origin in the horizontal and vertical directions, at time t seconds, is denoted by the variables x and y respectively. It is given that when $t = 0$, $x = 1$, $y = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 2$. The variables are related by the differential equations $\frac{\mathrm{d}y}{\mathrm{d}t} + y = \frac{\mathrm{e}^{-t}}{t-30}$ and $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \mathrm{e}^{-t}$.

(i) Using the substitution $w = y\mathrm{e}^t$, find y in terms of t. **[5]**

(ii) Find x in terms of t. **[5]**

(iii) Sketch the path travelled by the insect. **[2]**

7 The complex number z is given by $z^2 = 4 + 4\sqrt{3}\mathrm{i}$.

(i) Find z in exact cartesian form $x + \mathrm{i}y$, showing your workings clearly. **[4]**

(ii) Given that $(w^*)^3 = z^2$, where w^* is the conjugate of w, find w in the form $r\mathrm{e}^{\mathrm{i}\theta}$. **[4]**

(iii) The point representing the complex number v is obtained by a counter clockwise rotation of the point representing z^2 through one right angle about the point (0, 1) on the Argand diagram. By considering $z^2 - \mathrm{i}$, find v in the form $x + \mathrm{i}y$. **[3]**

8 **(a)** A sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = \cos x$ and

$$n(n+1)(u_n - u_{n+1}) = 2n \sin\frac{x}{2} \sin\left(n + \frac{1}{2}\right)x + \cos nx \quad \text{for } n \in \mathbb{Z}^+$$

where x is a constant.

Prove by the method of mathematical induction that

$$u_n = \frac{\cos nx}{n} \quad \text{for } n \in \mathbb{Z}^+.$$ **[6]**

(b) **(i)** Show that $\sum_{n=1}^{4N} \left(\frac{1}{n}\cos\frac{n\pi}{2}\right)$ where N is a positive integer, can be written as

$$\frac{1}{2}\sum_{n=1}^{2N} \frac{(-1)^n}{n}.$$ **[2]**

(ii) Use your result in **b(i)** and the series expansion of $\ln(1 + x)$ in MF15 to deduce the exact value of

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\cos\frac{n\pi}{2}\right).$$ **[3]**

9 The lines l_1 and l_2 meet at the point P. The line l_3 is coplanar with l_1 and l_2 and is perpendicular to l_1. Given that l_1 and l_2 are parallel to the vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, show that l_3 is parallel to the vector $\mathbf{b}-\left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$. **[3]**

The equations of l_1 and l_2 are now known to be $\mathbf{r}=\begin{pmatrix}3\\-5\\2\end{pmatrix}+t\begin{pmatrix}11\\10\\2\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}3\\-5\\2\end{pmatrix}+s\begin{pmatrix}17\\3\\4\end{pmatrix}$ respectively, where s and t are real parameters. Find the equation of the line l_3, given that l_3 also passes through P. **[2]**

The line l_4 has equation $\mathbf{r}=\begin{pmatrix}10\\-3\\1\end{pmatrix}+u\begin{pmatrix}-3\\4\\-1\end{pmatrix}$, where u is a real parameter. Determine if l_3 and l_4 are skew or intersecting. **[3]**

The line l_5 is perpendicular to both l_3 and l_4. Find the acute angle between l_5 and the plane containing l_1 and l_2. **[5]**

10 A curve C has parametric equations $x = 1 + \cos\theta$ and $y = 2\sin\theta$, where $0 \le \theta \le \pi$.

(i) Show that the equation of the tangent to C at the point with parameter θ is $y=2\left(-x\cot\theta+\cot\theta+\operatorname{cosec}\theta\right)$. **[3]**

(ii) The points P and Q on C have parameters $\theta=\dfrac{5\pi}{6}$ and $\dfrac{\pi}{6}$ respectively. The tangent at P meets the tangent at Q at the point R. Find the y-coordinate of R. **[3]**

(iii) The area of the region bounded by the tangent at P, the tangent at Q and the x-axis is denoted by A and the area of the region bounded by C and the x-axis is denoted by B. Find the exact value of the difference of A and B. **[8]**

——— END OF PAPER ———