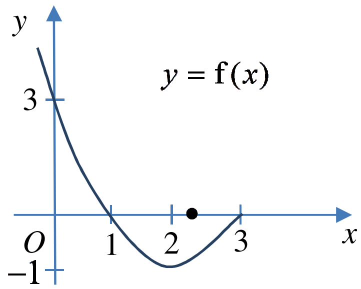
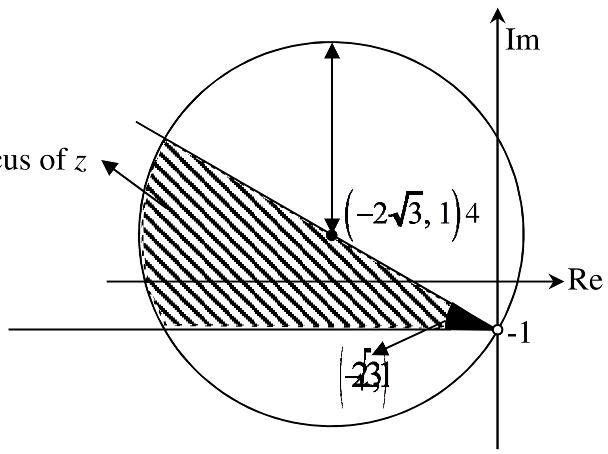
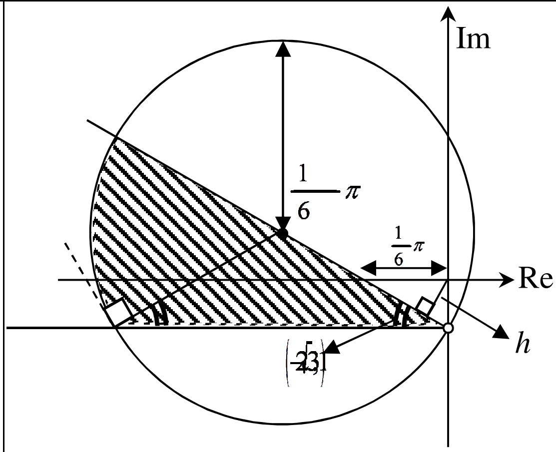
RAFFLES INSTITUTION

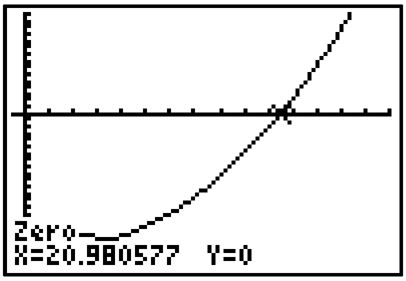
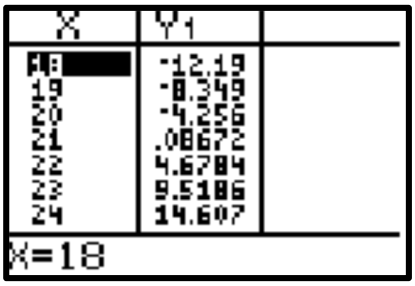
2014 YEAR 6 PRELIMINARY EXAMINATION

MATHEMATICS PAPER 2
(9740 / 2)
Higher 2

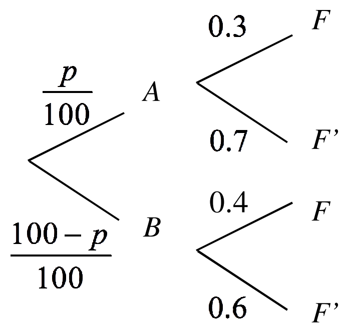
Qn. [Marks]	Solution
1(i) **[2]**	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{\sqrt{1-4x^2}}\right)=-\dfrac{1}{2}\left(1-4x^2\right)^{-\frac{3}{2}}\cdot(-4)(2x)$ $=\dfrac{4x}{\sqrt{\left(1-4x^2\right)^3}}$
1(ii) **[3]**	$\displaystyle\int \dfrac{x\sin^{-1}(2x)}{\sqrt{\left(1-4x^2\right)^3}}\ \mathrm{d}x$ $=\displaystyle\int \dfrac{4x}{\sqrt{\left(1-4x^2\right)^3}}\cdot\dfrac{1}{4}\sin^{-1}(2x)\ \mathrm{d}x$ $=\dfrac{1}{\sqrt{1-4x^2}}\cdot\dfrac{1}{4}\sin^{-1}(2x)\ -\displaystyle\int \dfrac{1}{\sqrt{1-4x^2}}\cdot\dfrac{1}{4}\dfrac{2}{\sqrt{1^2-(2x)^2}}\ \mathrm{d}x$ $=\dfrac{\sin^{-1}(2x)}{4\sqrt{1-4x^2}}-\dfrac{1}{4}\displaystyle\int \dfrac{2}{1^2-(2x)^2}\ \mathrm{d}x$ $=\dfrac{\sin^{-1}(2x)}{4\sqrt{1-4x^2}}-\dfrac{1}{8}\ln\left\|\dfrac{1+2x}{1-2x}\right\|+C$ or $\dfrac{\sin^{-1}(2x)}{4\sqrt{1-4x^2}}-\dfrac{1}{8}\ln\dfrac{1+2x}{1-2x}+C$
2(i) **[4]**	$\dfrac{2a}{x^2}>\dfrac{1+6a^2-3ax}{x}$ $\dfrac{2a}{x^2}-\dfrac{x+6a^2x-3ax^2}{x^2}>0$ $\dfrac{3ax^2-x-6a^2x+2a}{x^2}>0$ $\dfrac{(3ax-1)(x-2a)}{x^2}>0$ $x<0$ or $0<x<\dfrac{1}{3a}$ or $x>2a$
(ii) **[2]**	Replace x by $\sqrt{x}$, we have $0<\sqrt{x}<\dfrac{1}{3a}$ or $\sqrt{x}>2a$ or $\sqrt{x}<0$ (rejected since $\sqrt{x}\geq 0$)

Qn. [Marks]	Solution
	$0 < x < \dfrac{1}{9a^2}$ or $x > 4a^2$
3(i) **[1]**	
(ii) **[2]**	The greatest value of k is 2. Any horizontal line, $y = m,\ m \geq -1$ will cut the graph of f at one and only one point. Hence f is a one-one function and f^{-1} exists.
(iii) **[2]**	For gf to exist, $\mathrm{R_f} \subseteq \mathrm{D_g}$, ie $[-1, \infty) \subseteq \left(-\dfrac{a}{2}, \infty\right)$ Therefore $-1 > -\dfrac{a}{2} \Rightarrow a > 2$ Since $a \in \mathbb{Z}$, smallest value of a is 3.
(iv) **[2]**	$\mathrm{gf} = \mathrm{g}[\mathrm{f}(x)] = \ln\left[2\left(x^2 - 4x + 3\right) + 3\right] = \ln\left(2x^2 - 8x + 9\right)$, $x \leq 3$
(v) **[2]**	To find $\mathrm{R_{gf}}$: $(-\infty, 3] \xrightarrow{\mathrm{f}} [-1, \infty) \xrightarrow{\mathrm{g}} [0, \infty)$ i.e. $\mathrm{R_{gf}} = [0, \infty)$
4(i) **[4]**	
(ii) **[2]**	The least possible $\lvert z \rvert$ is given by the perpendicular distance from the origin to

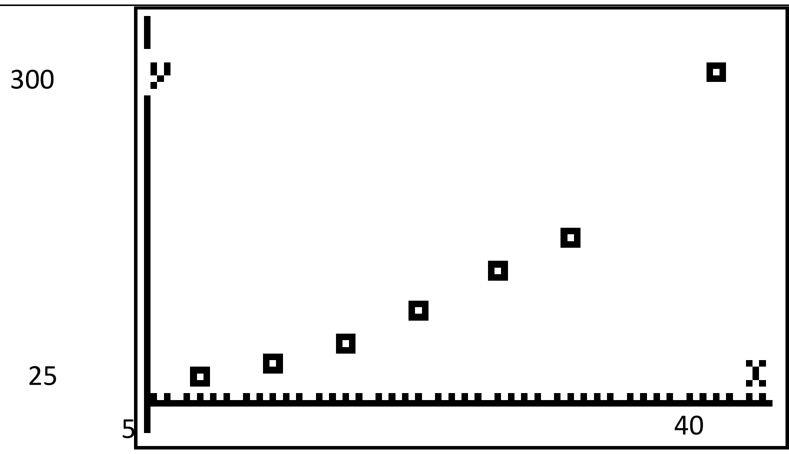
Qn. [Marks]	Solution
	 the half-line $\arg(z+\mathrm{i})=\frac{5}{6}\pi$, denoted by h as shown in the diagram. $h=\sqrt{3}\sin\frac{\pi}{6}$ or $1.\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$
(iii) [1]	$-4\sqrt{3}+3\mathrm{i}$
(iv) [2]	$0\le\arg\left(z+4\sqrt{3}+\mathrm{i}\right)<\frac{1}{6}\pi+\frac{1}{2}\pi \quad\Rightarrow\quad 0\le\arg\left(z+4\sqrt{3}+\mathrm{i}\right)<\frac{2}{3}\pi$
5(a) **[3]**	$9r^9=3\times 9r^3$ $r^6=3$, $\quad r\neq 0$ since c is non-zero The 22nd term is $9r^{21}=9\left(r^6\right)^3r^3=\pm 243\sqrt{3}$
[2]	$9r^{k-1}>10^{40}$ $3^{\frac{k-1}{6}}>\frac{10^{40}}{9}$ $k>1+\frac{6\ln\frac{10^{40}}{9}}{\ln 3}$ $k>492.02$ Least k is 493.
(b)(i) **[2]**	AP: $a=5.25$, $u_n=a+(n-1)d$ $u_8=6.65=5.25+(8-1)d$ $d=0.2$
(b)(ii) **[2]**	Runner A: Time taken to cover the first n laps, $S_n=\frac{n}{2}[2(5.25)+(n-1)(0.2)]$ $=5.15n+0.1n^2$ Runner B: Time taken to cover the first n laps, $S_n=\frac{6(1-0.99^n)}{1-0.99}=600(1-0.99^n)$

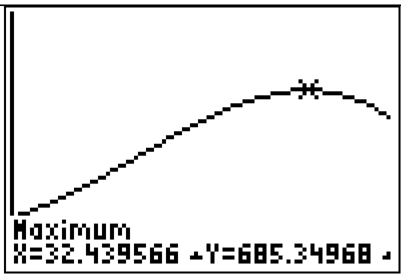
Qn. [Marks]	Solution
(b)(iii) **[2]**	Find n such that $\left\|S_{n(AP)} - S_{n(GP)}\right\| \geq 38$ $\left\|5.15n + 0.1n^2 - 600(1 - 0.99^n)\right\| \geq 38$ Using GC, sketch $y = \left\|5.15n + 0.1n^2 - 600(1 - 0.99^n)\right\| - 38$   When n =20, $\left\|5.15n + 0.1n^2 - 600(1 - 0.99^n)\right\| - 38 = -4.256 < 0$ When n = 21, $\left\|5.15n + 0.1n^2 - 600(1 - 0.99^n)\right\| - 38 = 0.8672 > 0$ The least number of laps is 21
6 **[3]** **[1]**	Assign every staff in the company a number from 1 to 500 by arranging their names in alphabetical order. As k = 10, randomly select one staff from the first 10 staff. If the 4th staff is chosen, then every 10th staff from the 4th staff on the list will be included in the sample, eg 4th , 14th , 24th ,…,494th staff will be included in the sample. The sampling frame (namelist etc) may exhibit a "cyclic" pattern, for instance every 10th staff in the list may just happen to be an "administrative" staff. Stratified or quota sampling.
7(i) **[2]**	Unbiased estimate of the population mean, $\bar{x} = 13.61$ Unbiased estimate of the population variance, $s^2 = (1.519566)^2 = 2.31$ (3 s.f)
(ii) **[1]**	μ denotes the population mean 2.4 km run times, in minutes, of Year 6 female students. $H_0: \mu = \mu_0$ vs $H_1: \mu < \mu_0$
(iii) **[3]**	Perform a 1-tail test at 5% significance level. Under H_0 , $\because n = 50$ is large $\bar{X} \sim N\left(\mu_0, \dfrac{s^2}{n}\right)$ approximately by Central Limit Theorem

Qn. [Marks]	Solution
	From the sample, $\bar{x}=13.61$, $n=50$ and $s=1.519566$ Using a z – test, for the null hypothesis to be rejected, ***p*- value** ≤ 0.05 $\mathrm{P}(\bar{X}<\bar{x})\leq 0.05$ $\mathrm{P}\left(Z<\dfrac{13.61-\mu_0}{\frac{1.519566}{\sqrt{50}}}\right)\leq 0.05$ $\dfrac{13.61-\mu_0}{\frac{1.519566}{\sqrt{50}}}\leq -1.64485$ $\mu_0 \geq 14.0$ (3 s.f)
8(i) **[1]**	Number of ways to seat them with no restriction = 10! = 3 628 800
(ii) **[3]**	Number of ways that the 2 students are seated next to each other $= {}^8C_1\times 2!\times 8!$ or $2\left[{}^8C_3\times 4!\times 2!\times 5!\right]$ =645 120 Total number of arrangements = 3 628 800 – 645 120 = 2 983 680 **OR** <u>Case 1: The 2 students seated on separate rows</u> Number of ways = ${}^5C_1\times {}^5C_1\times 8!\times 2=2016000$ <u>Case 2: The 2 students seated on same row</u> Number of ways = ${}^8P_3\times {}^4P_2\times 5!\times 2=967860$ **OR** <u>Case 1: 1st student seated at one of the four corners</u> Number of ways = $4\times 8\times 8!=1290240$ <u>Case 2: 1st student not seated at one of the four corners</u> Number of ways = $6\times 7\times 8!=1693440$

Qn. [Marks]	Solution
(iii) **[4]**	<u>Case 1: 3 students are seated in front row with remaining 5 taller students at the back row</u> Number of ways = ${}^{5}C_{3} \times 3! \times 5!$ = 7200. <u>Case 2: 4 students are seated in front row with remaining 4 taller students at the back row</u> Number of ways = $\left({}^{5}C_{4} \times 4!\right)^{2} = 14400$ **OR** Treating empty seats as distinct and we have 4 students per row, thus number of ways = $5! \times 5! = 14400$ <u>Case 3: 5 students are seated in front row with remaining 3 taller students at the back row</u> Number of ways = 7200 (similar to Case 1). $\therefore$ Total number of possible seating arrangement = 7200(2) + 14400 = 28800
9(i) **[5]**	Let A and B be the event that a student takes Math A and Math B respectively. Let F be the event that a student failed the paper that he/she sat for.  $\mathrm{P}(F) = \frac{0.3p}{100} + 0.4\left(\frac{100-p}{100}\right) = \frac{40-0.1p}{100}$ $\mathrm{P}(F') = 1 - \frac{40-0.1p}{100} = \frac{60+0.1p}{100}$

Qn. [Marks]	Solution
	P(exactly one out of 2 failed) = P(one failed and one passed) $= {}^2C_1\left(\dfrac{40-0.1p}{100}\right)\left(\dfrac{60+0.1p}{100}\right)$ $=\dfrac{2}{10000}\left(2400-4p+6p-0.01p^2\right)$ $=0.0002(2400-2p-0.01p^2)$ (shown)
(ii) **[3]**	P(both take Math A | exactly one out of 2 failed) $=\dfrac{\text{P(one takes Math A and failed and the other takes Math A and passed)}}{\text{P(exactly one out of 2 failed)}}$ $=\dfrac{{}^2C_1\left(\dfrac{0.3p}{100}\right)\left(\dfrac{0.7p}{100}\right)}{0.0002(2400-2p-0.01p^2)}$ $=\dfrac{0.21p^2}{2400-2p-0.01p^2}=\dfrac{7}{48}$ Hence, $10.08p^2=16800-14p-0.07p^2$ $10.15p^2+14p-16800=0$ $p=40$ or $\dfrac{-1200}{29}$ (n.a since $p\geq 0$)
10(i) **[2]**	Let X represent number of large air bubbles in a glass panel $X\sim \text{Po}(0.2)$ Let Y represent number of small air bubbles in a glass panel $Y\sim \text{Po}(1.8)$ Since X and Y are independent, $X+Y\sim \text{Po}(2.0)$ $P(X+Y\leq 2)=0.677$ (3 s.f)
(ii) **[3]**	$P(X\geq 1 \mid X+Y\leq 2)$ $=\dfrac{P(X=1,Y=0,1)+P(X=2,Y=0)}{P(X+Y\leq 2)}$ $=\dfrac{(0.16375)(0.46284)+(0.016375)(0.16530)}{0.67667}$ $=0.116$ (3 s.f)
(iii) **[2]**	Let S represent the number of glass panels out of 12 containing at most 2 air bubbles. $S\sim B(12,0.67667)$ $P(S\geq 7)=1-P(S\leq 6)=0.842$ (3 s.f)

Qn. [Marks]	Solution
(iv) **[3]**	Let T represent the total number of air bubbles in 12 glass panels $T \sim \text{Po}(24)$ Since $\lambda = 24$ is large, $T \sim N(24, 24)$ approximately $P(T < 30)$ $= P(T < 29.5)$ using continuity correction $= 0.869$ (3 s.f.)
11(i) **[1]**	
(ii) **[2]**	$y = -46.016 + 7.5180x = -46.0 + 7.52x$ (3 s.f) When $x = 5$, $y = -8.43$. (3 s.f)
(iii) **[2]**	1. The number of bottles sold, y, cannot have a negative value. 2. From the scatter diagram, as x increases, y increases at an increasing rate, which does not fit a linear model. Hence the linear model is not suitable.
(iv) **[2]**	$r = 0.995$ (3 s.f) Since the value of r is close to 1, there is a strong positive linear correlation between y and x^2.
(v) **[2]**	$y = 14.898 + 0.17128x^2$ $a = 14.9$, $b = 0.171$ (3 s.f)
[3]	Let P be the profit . $P = \left[\left(\frac{100-x}{100}\right) \times 20 - 10\right] y$ --- (1) Substitute $y = 14.898 + 0.17128x^2$ into (1)

Qn. [Marks]	Solution
	 By curve sketching, the hair stylist will maximize his profit when $x = 32\%$. (nearest whole number)
12(i) **[3]**	Let C be the mass of 6 chicken eggs and D be the mass of 4 duck eggs. $C \sim N\left(6(50), 6(2^2)\right)$ and $D \sim N\left(4(65), 4(3^2)\right)$ $C \sim N(300,\ 24)$ and $D \sim N(260,\ 36)$ $C_1 + C_2 - 2D \sim N\left(300+300-2(260),\ 24+24+2^2(36)\right)$ $C_1 + C_2 - 2D \sim N(80,\ 192)$ $P(C_1 + C_2 - 2D > 100) = 0.0745$ (3s.f)
(ii) **[5]**	Let X be the mass of 1 chicken egg. $X \sim N(50,\ 2^2)$ $P(\lvert X-50\rvert < 2(2)) = P\left(\left\lvert \frac{X-E(X)}{2} \right\rvert < 2\right)$ $= P(-2 < Z < 2)$ $= 0.954499876 = 0.95450$ (5s.f) Let W be the number of chicken eggs out of 60 that are not "good". $W \sim B(60,\ 1-0.95450)$ $\therefore W \sim B(60,\ 0.04550)$ Since $n = 60$ large and $p = 0.04550$ small such that $np = 2.73 < 5$, $W \sim Po(2.73)$ approximately $P(\text{at least 55 eggs are "good" out of 60}) = P(W \le 5) = 0.941$ (3s.f)
(iii) **[2]**	$D \sim N(260,\ 36)$ $M \sim N\left(260,\ \frac{36}{4}\right)$ $P(M > 265) = 0.0478$ (3s.f)
(iv) **[2]**	$M \sim N\left(260,\ \frac{36}{n}\right)$

Qn. [Marks]	Solution
	$P(M > 265) = P\left(\dfrac{M-260}{6/\sqrt{n}} > \dfrac{265-260}{6/\sqrt{n}}\right)$ $= P\left(Z > \dfrac{5\sqrt{n}}{6}\right) \approx 0$ for large n. Alternatively, $n \to \infty \Rightarrow \dfrac{36}{n} \to 0$ Since the variance decreases to approximately 0 and the mean is 260, $P(M > 265) \to 0$.