Suggested Solutions to 2014 SH2 H2 Mathematics Preliminary Examination Paper 2

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Let V denote the volume of the container.

$$2\left(\frac{1}{2}r^2\theta\right)+2(r)(3r)+r\theta(3r)=10$$

$$4r^2\theta+6r^2=10$$

$$\theta=\frac{10-6r^2}{4r^2}$$

$$=\frac{5-3r^2}{2r^2}$$

$$V=\frac{1}{2}r^2\theta(3r)$$

$$=\frac{3}{2}r^3\left(\frac{5-3r^2}{2r^2}\right)$$

$$=\frac{3}{4}\left(5r-3r^3\right)$$

$$\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{3}{4}\left(5-9r^2\right)$$

For stationary V, $\frac{\mathrm{d}V}{\mathrm{d}r}=0$

$$5-9r^2=0$$

$$r^2=\frac{5}{9}$$

$$r=\frac{\sqrt{5}}{3}$$

Corresponding value of θ

$$=\frac{5-3\left(\frac{5}{9}\right)}{2\left(\frac{5}{9}\right)}=3$$

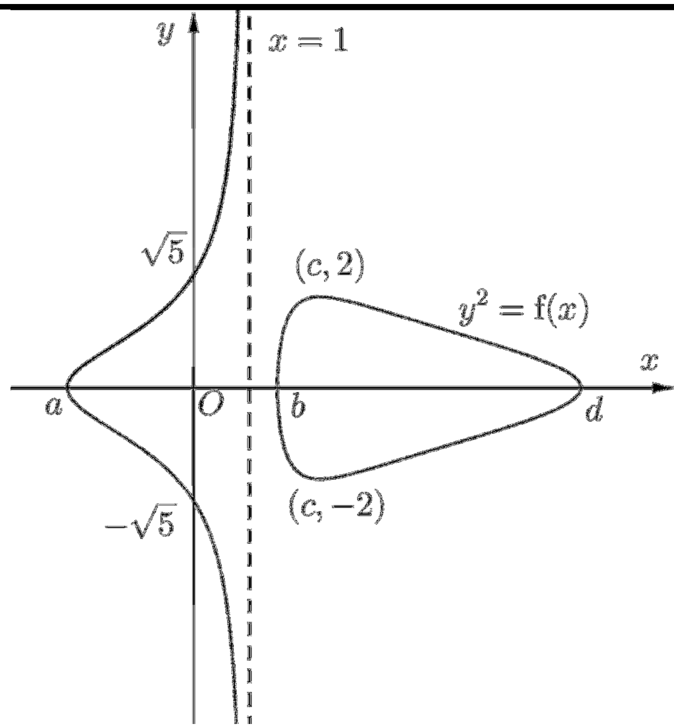
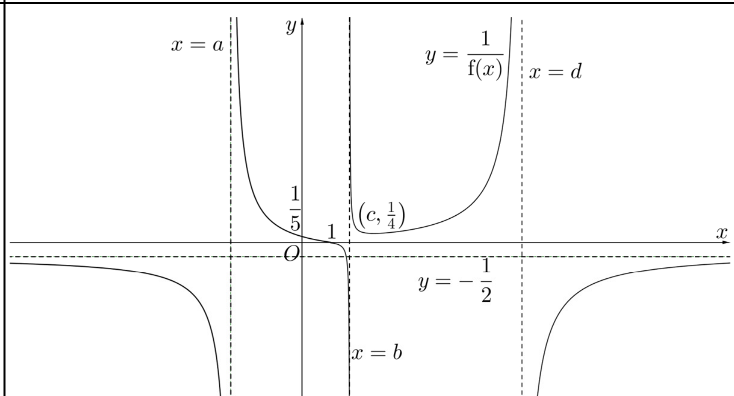
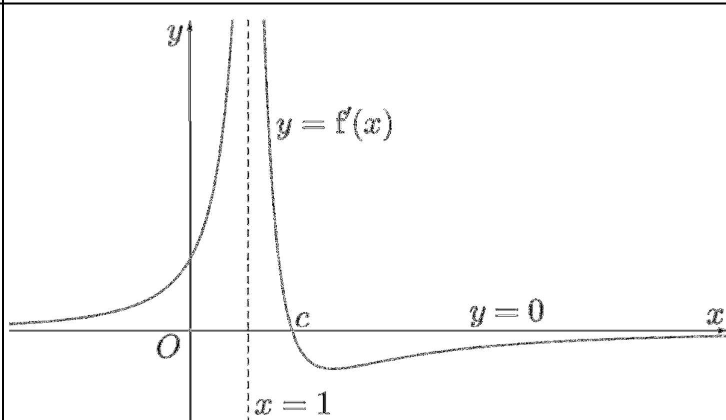
<u>Second Derivative Test</u>

$$\left.\frac{\mathrm{d}^2V}{\mathrm{d}r^2}\right|_{r=\frac{\sqrt{5}}{3}}=\left.\frac{3}{4}(-18r)\right|_{r=\frac{\sqrt{5}}{3}}=-\frac{9\sqrt{5}}{2}<0$$

<u>First Derivative Test</u>

r	$\frac{\sqrt{5}}{3}-0.1$	$\frac{\sqrt{5}}{3}$	$\frac{\sqrt{5}}{3}+0.1$
$\frac{\mathrm{d}V}{\mathrm{d}r}$	$0.9387>0$	0	$-1.0737<0$

Hence, V is a maximum when $r=\frac{\sqrt{5}}{3}$.

2 (a)	y $x = 1$ $\sqrt{5}$ $(c, 2)$ $y^2 = \mathrm{f}(x)$ x a O b d $(c, -2)$ $-\sqrt{5}$
2 (b)	$x = a$ y $y = \frac{1}{\mathrm{f}(x)}$ $x = d$ $\frac{1}{5}$ $(c, \frac{1}{4})$ 1 x O $y = -\frac{1}{2}$ $x = b$
2 (c)	y $y = \mathrm{f}'(x)$ c $y = 0$ x O $x = 1$

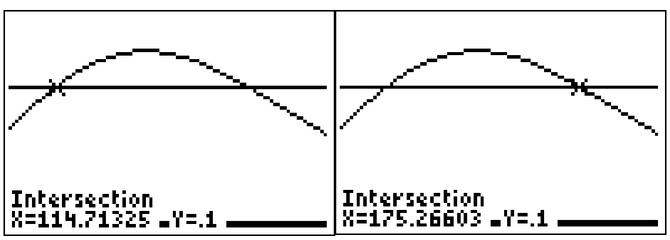
3 (a)	<u>Method 1: Expressing z in the form $x + y\text{i}$</u> Let $z = x + y\text{i}$. $\left\lvert \frac{2\text{i} - z^*}{z} - 1 \right\rvert^2 - z = \text{i}$ $\left\lvert \frac{2\text{i} - (x + y\text{i})^* - (x + y\text{i})}{x + y\text{i}} \right\rvert^2 - (x + y\text{i}) = \text{i}$ $\left\lvert \frac{2\text{i} - 2x}{x + y\text{i}} \right\rvert^2 - (x + y\text{i}) = \text{i}$ $\frac{4x^2 + 4}{x^2 + y^2} - x - y\text{i} = \text{i}$ Comparing real and imaginary parts, $\frac{4x^2 + 4}{x^2 + y^2} - x = 0$ and $-y = 1$. $\therefore y = -1$ $\frac{4(x^2 + 1)}{x^2 + 1} - x = 0 \Rightarrow 4 - x = 0$ $x = 4$. Thus, $z = 4 - \text{i}$.
	<u>Method 2: Observing that the modulus of a complex number is real</u> Let $z = x + y\text{i}$. Since $\left\lvert \frac{2\text{i} - z^*}{z} - 1 \right\rvert^2 \in \mathbb{R}$, $-y = 1 \Rightarrow y = -1$. Therefore $z = x - \text{i}$. Hence, $\left\lvert \frac{2\text{i} - (x + \text{i})}{x - \text{i}} - 1 \right\rvert^2 - x + \text{i} = \text{i}$ $\left\lvert \frac{\text{i} - x}{x - \text{i}} - 1 \right\rvert^2 - x = 0$ $\lvert -1 - 1 \rvert^2 - x = 0$ $x = 4$ Hence $z = 4 - \text{i}$.

3 (b) (i)	$p=-\sqrt{3}+\mathrm{i}=2\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}$ $q=-4\mathrm{i}=4\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}$
3 (b) (ii)	$\dfrac{p^{10}}{q^{5}}=\dfrac{2^{10}\mathrm{e}^{\mathrm{i}\frac{50\pi}{6}}}{4^{5}\mathrm{e}^{-\mathrm{i}\frac{5\pi}{2}}}=\mathrm{e}^{\mathrm{i}\left(\frac{50\pi}{6}+\frac{5\pi}{2}\right)}=\mathrm{e}^{\mathrm{i}\left(\frac{65\pi}{6}\right)}$ $\dfrac{p^{10}}{q^{5}}+\dfrac{q^{5}}{p^{10}}=\mathrm{e}^{\mathrm{i}\frac{65\pi}{6}}+\dfrac{1}{\mathrm{e}^{\mathrm{i}\frac{65\pi}{6}}}$ $=\mathrm{e}^{\mathrm{i}\frac{65\pi}{6}}+\mathrm{e}^{-\mathrm{i}\frac{65\pi}{6}}$ $=\cos\dfrac{65\pi}{6}+\mathrm{i}\sin\dfrac{65\pi}{6}+\cos\left(-\dfrac{65\pi}{6}\right)+\mathrm{i}\sin\left(-\dfrac{65\pi}{6}\right)$ $=2\cos\left(\dfrac{65\pi}{6}\right)$ $=-2\cos\left(\dfrac{\pi}{6}\right)=-\sqrt{3}$

4 (i)	<u>Method 1</u> The line passing through C perpendicular to p_1 has a vector equation $\mathbf{r}=\begin{pmatrix}-3\\5\\0\end{pmatrix}+\mu\begin{pmatrix}1\\-3\\1\end{pmatrix},\ \mu\in\mathbb{R}.$ Substitute $x=-3+\mu,\ y=5-3\mu,\ z=\mu$ into $x-3y+z=4$, $\begin{aligned}(-3+\mu)-3(5-3\mu)+\mu&=4\\11\mu-18&=4\\\mu&=2\end{aligned}$ $\overrightarrow{OM}=\begin{pmatrix}-3\\5\\0\end{pmatrix}+2\begin{pmatrix}1\\-3\\1\end{pmatrix}=\begin{pmatrix}-1\\-1\\2\end{pmatrix}$
	<u>Method 2</u> A vector equation of p_1 is $\mathbf{r}\cdot\mathbf{n_1}=4$ where $\mathbf{n_1}=\begin{pmatrix}1\\-3\\1\end{pmatrix}$ Take any point A on p_1, e.g. $(1,-1,0)$, then $\overrightarrow{CM}$ is the projection vector of $\overrightarrow{CA}$ onto $\mathbf{n_1}$. $\overrightarrow{CA}=\overrightarrow{OA}-\overrightarrow{OC}=\begin{pmatrix}1\\-1\\0\end{pmatrix}-\begin{pmatrix}-3\\5\\0\end{pmatrix}=\begin{pmatrix}4\\-6\\0\end{pmatrix}$ $\begin{aligned}\overrightarrow{CM}&=\left(\overrightarrow{CA}\cdot\hat{\mathbf{n}}_1\right)\hat{\mathbf{n}}_1\\&=\left[\frac{\begin{pmatrix}4\\-6\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-3\\1\end{pmatrix}}{\sqrt{1^2+(-3)^2+1^2}}\right]\frac{1}{\sqrt{1^2+(-3)^2+1^2}}\begin{pmatrix}1\\-3\\1\end{pmatrix}\\&=\begin{pmatrix}2\\-6\\2\end{pmatrix}\end{aligned}$ $\overrightarrow{OM}=\overrightarrow{OC}+\overrightarrow{CM}=\begin{pmatrix}-3\\5\\0\end{pmatrix}+\begin{pmatrix}2\\-6\\2\end{pmatrix}=\begin{pmatrix}-1\\-1\\2\end{pmatrix}$

4 (ii)	$\overrightarrow{OM}=\dfrac{\overrightarrow{OC}+\overrightarrow{OQ}}{2}$ $\overrightarrow{OQ}=2\overrightarrow{OM}-\overrightarrow{OC}$ $=2\begin{pmatrix}-1\\-1\\2\end{pmatrix}-\begin{pmatrix}-3\\5\\0\end{pmatrix}=\begin{pmatrix}1\\-7\\4\end{pmatrix}$ The coordinates of Q are $(1,-7,4)$.
4 (iii)	A normal vector to p_2: $\mathbf{n}_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$. A Cartesian equation of p_2 is $y=5$ (or a vector equation of p_2 is $\mathbf{r}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}=5$) Solve $x-3y+z=4$ and $y=5$ simultaneously, we have $x=19-\lambda,\ y=5,\ z=\lambda$. A vector equation of l is $\mathbf{r}=\begin{pmatrix}19\\5\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\0\\1\end{pmatrix}, \lambda\in\mathbb{R}$. or $\mathbf{r}=\begin{pmatrix}0\\5\\19\end{pmatrix}+\lambda\begin{pmatrix}-1\\0\\1\end{pmatrix}, \lambda\in\mathbb{R}$
4 (iv)	$Q(1,-7,4)$ lies on p_3. Two of the vectors parallel to p_3 are $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}19\\5\\0\end{pmatrix}-\begin{pmatrix}1\\-7\\4\end{pmatrix}=\begin{pmatrix}18\\12\\-4\end{pmatrix}=2\begin{pmatrix}9\\6\\-2\end{pmatrix}$. A normal vector to p_3 is $\begin{pmatrix}-1\\0\\1\end{pmatrix}\times\begin{pmatrix}9\\6\\-2\end{pmatrix}=\begin{pmatrix}-6\\7\\-6\end{pmatrix}$. $\begin{pmatrix}1\\-7\\4\end{pmatrix}\cdot\begin{pmatrix}-6\\7\\-6\end{pmatrix}=-79$, thus a Cartesian equation of p_3 is $-6x+7y-6z=-79$.

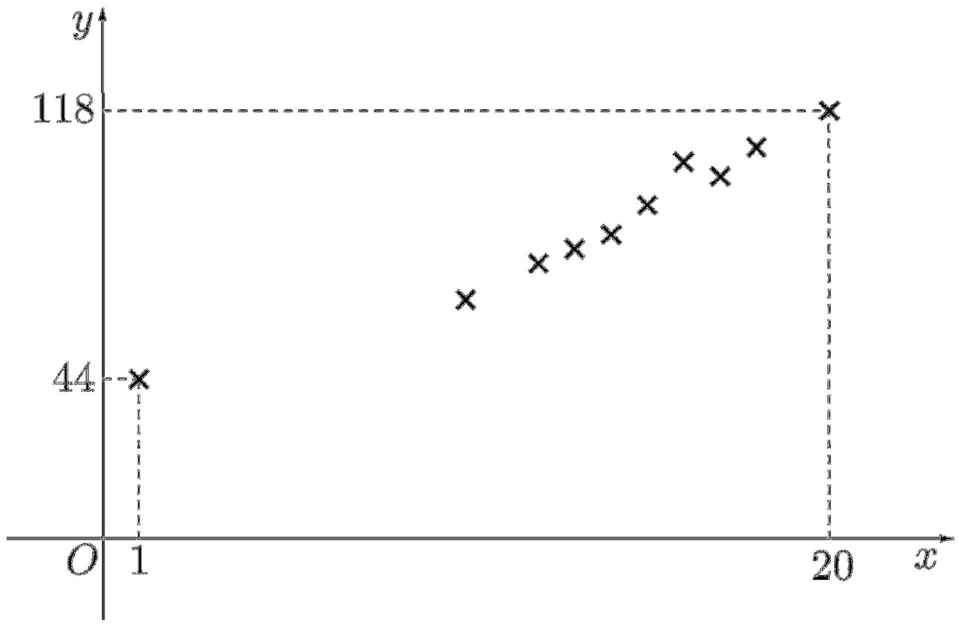
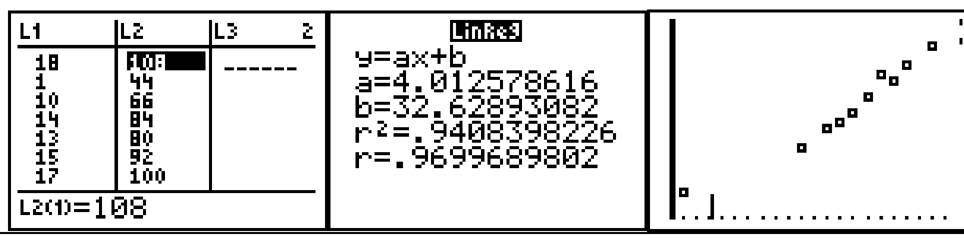
5 (a)	Number of ways $= \underbrace{{}^{10}\mathrm{P}_8}_{\text{Permutate 10 distinct seats between 8 distinct people}}$ or $\underbrace{{}^{10}\mathrm{C}_8}_{\text{Select 8 of the 10 seats}} \times \overbrace{8!}^{\text{Arrange the 8 people in the 8 chosen seats}}$ or $\dfrac{\overbrace{10!}^{\text{Arrange 8 people and 2 seats}}}{\underbrace{2!}_{\text{2 identical seats}}}$ $=1814400$
5 (b)	Number of ways $= \underbrace{4!}_{\text{Arranging the other 4 people to form barriers}} \times \underbrace{{}^{5}\mathrm{P}_2}_{\text{Permutating 2 of the 5 "seats" formed between Derrick and the 3 girls "unit"}} \times \underbrace{3!}_{\text{Internal arrangement of the 3 girls}}$ $=2880$
6 (a)	Obtain a list of names of all students in the graduating cohort in some order from the school administration, say by registration number or identification number. Select a number k randomly from 1 to $\frac{550}{50}=11$ by using a random number generator and select the k^{th} student and every 11^{th} student thereafter on the list, until a sample of 50 students is obtained.
6 (b)	Put up the survey online for all students in the graduating cohort and collect the responses from the first 50 students who respond and satisfy the following criteria and quota: <table><tr><td></td><td>Arts Stream</td><td>Science Stream</td></tr><tr><td>Quota</td><td>25</td><td>25</td></tr></table>
6 (last part)	By using quota sampling, the admissions office can choose to categorise all the graduating students, say by subject combination or stream, and select a certain number of students to <u>represent</u> each category, as this may affect their analysis for the students' interests in specific university courses and career paths. Furthermore, even though it should not be difficult to obtain a school register for all the graduating cohort from the school administration to carry out systematic sampling, it can be time consuming and inefficient to <u>identify</u> and survey all the students chosen from the random selection, as they may likely be from different classes.

7 (a)	**Accepted answers:** The mean number of students who tap their fingers at the fingerprint sensor in any second (or unit interval of time) during the one-minute interval must be constant, or equivalently, the events that students tap their fingers at the fingerprint sensor occur at a constant mean rate throughout the one-minute interval. The event that a student taps his/her finger at the fingerprint sensor in any period of time is independent of another student tapping his/her finger at the same fingerprint sensor in another non-overlapping period of time during the one-minute interval. The events that students tap their fingers at the fingerprint sensor occur randomly throughout the one-minute interval.
7 (b)	Let X be the number of students who tap their fingers at the fingerprint sensor in a one-minute period. Then $X \sim \text{Po}(4.2)$. Hence required probability = $\text{P}(X \geq 3)$ $= 1 - \text{P}(X \leq 2)$ $= 0.78976$ $= 0.790$ (correct to 3 significant figures)
7 (c)	Let Y be the number of students who tap their fingers at the fingerprint sensor in a t-seconds period. Then $Y \sim \text{Po}\left(\frac{4.2t}{60}\right)$ i.e. $Y \sim \text{Po}(0.07t)$. Hence an equation for t would be $\frac{\mathrm{e}^{-0.07t}(0.07t)^{10}}{10!} = 0.1$.  By sketching the graphs of $y = \frac{\mathrm{e}^{-4.2x}(4.2x)^{10}}{10!}$ and $y = 0.1$ using the GC (as shown above), we have $t = 115$ or $t = 175$

8 (a) (i)	$\mathrm{P}(X=34 \text{ or } 35)=\mathrm{P}(X=34)+\mathrm{P}(X=35)$ $=0.075128$ $=0.0751$ (to 3 s.f.)
8 (a) (ii)	$\mathrm{P}\left(X=34 \text{ or } 35 \mid X \le \mathrm{E}(X)\right)$ $=\mathrm{P}\left(X=34 \text{ or } 35 \mid X \le 64\times\frac{5}{8}\right)$ $=\frac{\mathrm{P}(X=34 \text{ or } 35)}{\mathrm{P}(X\le 40)}$ $=\frac{0.075128}{0.547067}$ $=0.13732$ $=0.137$ (to 3 s.f.)
8 (b)	Since $n=64>30$ is large, p is large enough such that $np=40>5$ and $n(1-p)=24>5$, $X\sim \mathrm{N}(40,15)$ approximately. $\mathrm{P}(30<X\le 45)=\mathrm{P}(30.5<X<45.5)$ (by continuity correction) $=0.915$ (correct to 3sf)
9 (a)	Required probability = $\frac{\binom{15}{3}\binom{25}{5}}{\binom{40}{8}}$ = 0.314339 = 0.314 (3 sf) OR Required probability = $\left(\frac{15}{40}\right)\left(\frac{14}{39}\right)\left(\frac{13}{38}\right)\left(\frac{25}{37}\right)\left(\frac{24}{36}\right)\left(\frac{23}{35}\right)\left(\frac{22}{34}\right)\left(\frac{21}{33}\right)\times\frac{8!}{3!5!}$ = 0.314
9 (b) (i)	P(one gets the flu) $=(0.85)p+0.15(0.91)$ $=(0.85)(0.23)+(0.15)(0.91)$ $=0.332$ or $\frac{83}{250}$
9 (b) (ii)	$\mathrm{P}\left(\begin{matrix}\text{exactly one of two people took the}\\ \text{flu vaccine} \mid \text{both of them get the flu}\end{matrix}\right)$ $=\frac{(0.85\times0.23)\times(0.91\times0.15)\times 2!}{(0.332)^2}$ $=0.48421=0.484(3\text{sf})$

10 (i)	$\overline{C} \sim N\left(2.2, \frac{0.5^2}{n}\right)$ $P\left(\overline{C} > 2.35\right) = 0.0502$ $P\left(Z \le \frac{2.35-2.2}{0.5/\sqrt{n}}\right) = 0.9498$ $\frac{2.35-2.2}{0.5/\sqrt{n}} = 1.643$ $n = 29.99$ Hence, $n = 30$.
10 (ii)	Let X be the price of one chicken. $X \sim \mathrm{N}\left(6.6, 1.5^2\right)$ Let Y be the price of one turkey. $Y \sim \mathrm{N}\left(55, 10.5^2\right)$ $X_1 + X_2 + \cdots + X_{17} \sim \mathrm{N}\left(112.2, 38.25\right)$ $2Y \sim \mathrm{N}\left(110, 441\right)$ $X_1 + X_2 + \cdots + X_{17} - 2Y \sim \mathrm{N}\left(2.2, 479.25\right)$ $\mathrm{P}\left(X_1 + X_2 + \cdots + X_{17} > 2Y\right)$ $= \mathrm{P}\left(X_1 + X_2 + \cdots + X_{17} - 2Y > 0\right)$ $= 0.540$
10 (iii)	The weights (or costs) of all poultry (chickens and turkeys) are independent of one another.

11 (i)	Unbiased estimate of the mean of X $= \frac{19}{10} + 100 = \frac{1019}{10}$ (or 101.9) Unbiased estimate of the variance of X $= \frac{1}{9}\left[159 - \frac{(19)^2}{10}\right] = \frac{1229}{90}$ (or 13.7)
11 (ii)	Assume that X follows a normal distribution. Let μ denote the mean of X. H_0: $\mu = 100$ vs H_1: $\mu > 100$
11 (iii)	Level of significance: 5% (upper-tailed) Under H_0, test statistic: $T = \frac{\bar{X} - 100}{s/\sqrt{10}} \sim t_9$. Using GC, p-value = 0.0692 (> 0.05). OR $t_{calc} = 1.63$ (< 1.83) We do not reject H_0 and conclude that there is insufficient evidence at 5% significance level to indicate that the company is underestimating the mean calorie claimed in a mini-size packet of its chocolate chip cookies.
11 (iv)	There is a probability of $\frac{\alpha}{100}$ that we will reject the manufacturer's claim that the mean number of calories of a mini-size packet of its chocolate chip cookies is 100 when it is in fact true.
11 (last part)	H_0: $\mu = 100$ H_1: $\mu \neq 100$ Level of significance: 2% (two-tailed) Under H_0, test statistic: $Z = \frac{\bar{X} - 100}{\sqrt{13/10}} \sim N(0,1)$. For the manufacturer's claim to be incorrect, we will reject H_0. Thus, $\frac{\bar{x} - 100}{\sqrt{13/10}} < -2.32635$ or $\frac{\bar{x} - 100}{\sqrt{13/10}} > 2.32635$ $\Rightarrow \bar{x} < 97.3475$ or $\bar{x} > 102.6524$ $\Rightarrow \bar{x} < 97.3$ or $\bar{x} > 102.7$ OR $\Rightarrow \bar{x} \leq 97.3$ or $\bar{x} \geq 102.7$ OR $\Rightarrow \bar{x} < 97.4$ or $\bar{x} > 102.6$

12 (i)	 
12 (ii)	r-value = 0.970 (3 sig. fig.)
12 (iii)	The rate of increase is not uniform OR The points appear to follow a curvilinear trend or a non-linear model.
12 (iv)	The regression line to be used is the y on x line. This is because the weight of a chick is dependent on its age.
12 (v)	(A) $y = ax + b$: r-value = 0.970 (B) $y = cx^2 + d$: r-value = 0.987 (C) $y = e\sqrt{x} + f$: r-value = 0.914 As x increases, y increases at an increasing rate **OR** (B) has r-value closest to 1, (B) is the best model.
12 (vi)	Regression line of y on x^2 is $y = 0.18814x^2 + 47.614 = 0.188x^2 + 47.6$ When $x = 8$, $y = 0.18814(8)^2 + 47.614$ $= 59.654$ $= 59.7$ Since $x = 8$ lies in the range of values of x, [1, 20] and the absolute value of r, 0.987 is close to 1, which indicates a strong positive **linear** relationship between x^2 and y, the estimate is reliable.