2014 H2 MATH (9740) JC2 PRELIMINARY EXAM PAPER 2 SOLUTION

Qn	Solution
1	**APGP**

Loan amount = $0.95(350000) = 332500$

Interest rate per month = 0.25%

Month	Start of month	End of month
1	$(332500)1.0025$	$(332500)1.0025 - x$
2	$[(332500)1.0025 - x]1.0025$ $= 332500(1.0025^2) - 1.0025x$	$332500(1.0025^2) - 1.0025x - x$
3	$\left(332500(1.0025^2) - 1.0025x - x\right)1.0025$ $= 332500(1.0025^3) - 1.0025^2 x - 1.0025x$	$= 332500(1.0025^3) - 1.0025^2 x$ $-1.0025x - x$

At the end of the nth month, the outstanding amount owed

$= 332500(1.0025^n) - 1.0025^{n-1}x - ... - 1.0025x - x$

$= 332500(1.0025^n) - x(1 + 1.0025 + ... + 1.0025^{n-1})$

$= 332500(1.0025)^n - x\dfrac{1.0025^n - 1}{1.0025 - 1}$

$= 332500(1.0025)^n - 400x\left(1.0025^n - 1\right)$ (shown)

To pay off the entire loan by the 30th year, the outstanding amount owed at the end of the 360th **month** must be less than or equal to 0.

$332500(1.0025)^{360} - 400x\left(1.0025^{360} - 1\right) \leq 0$

$400x\left(1.0025^{360} - 1\right) \geq 332500(1.0025)^{360}$

$x \geq \dfrac{332500(1.0025)^{360}}{400\left(1.0025^{360} - 1\right)}$

$x \geq 1401.83$

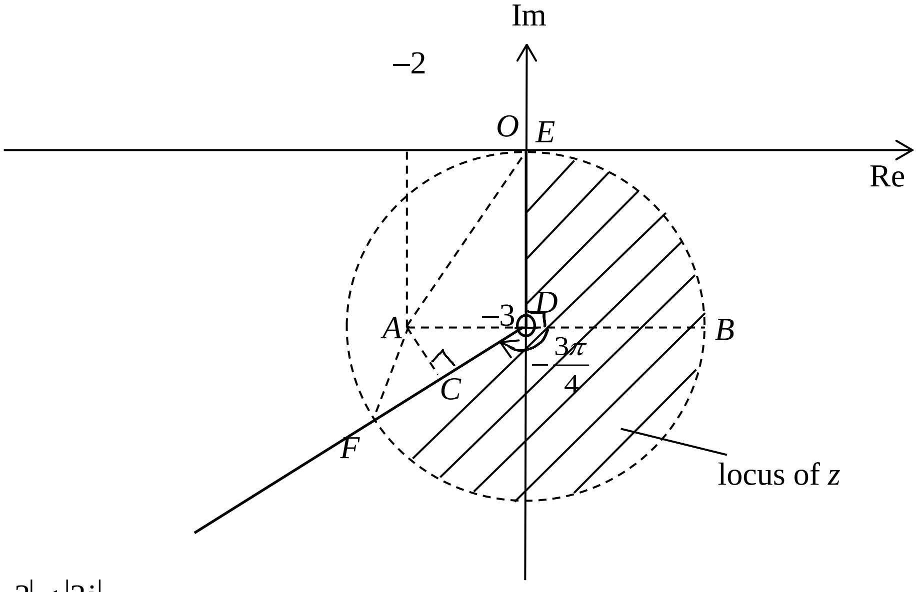
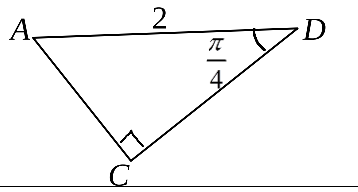
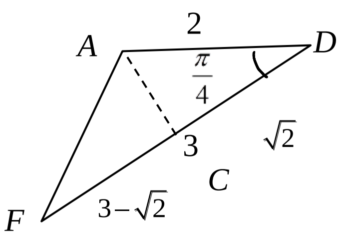
The least monthly installment is $1402.

This is due to the effect of compounded interest. As the length of the loan increases, the effect of compounded interest will accumulate. He will have to pay much more than the principal amount as the length of the repayment period increases.

The longer the loan, the more compounded interest is added and hence Andrew is paying the interest after 1 year and has not repaid much of the principal loan amount.

Qn	Solution
2	**Vectors**
(i)	$\begin{pmatrix}-1\\3\\1\end{pmatrix}\cdot\begin{pmatrix}3\\1\\-1\end{pmatrix}=-1\neq 0$ Therefore, l_1 and l_2 are not perpendicular.
(ii)	$\begin{pmatrix}1\\4\\1\end{pmatrix}+\lambda\begin{pmatrix}-1\\3\\1\end{pmatrix}=\begin{pmatrix}5\\2\\-1\end{pmatrix}+\mu\begin{pmatrix}3\\1\\-1\end{pmatrix}$ $1-\lambda=5+3\mu \Rightarrow -\lambda-3\mu=4$--- (1) $4+3\lambda=2+\mu \Rightarrow 3\lambda-\mu=-2$--- (2) $1+\lambda=-1-\mu \Rightarrow \lambda+\mu=-2$--- (3) Using GC, $\lambda=-1, \mu=-1$ $\Rightarrow \overrightarrow{OP}=\begin{pmatrix}2\\1\\0\end{pmatrix}$ $\therefore$ coordinates of P are $(2,1,0)$
(iii)	Since all the lines in π are perpendicular to AB, $\mathbf{n}$, the normal vector of π, is parallel to $\overrightarrow{AB}$. $\overrightarrow{AB}=\begin{pmatrix}5\\2\\-1\end{pmatrix}-\begin{pmatrix}1\\4\\1\end{pmatrix}=\begin{pmatrix}4\\-2\\-2\end{pmatrix}$ $\therefore \mathbf{n}=\begin{pmatrix}-2\\1\\1\end{pmatrix}$ π passes through M, the midpoint of A and B. $\overrightarrow{OM}=\frac{1}{2}\left[\begin{pmatrix}5\\2\\-1\end{pmatrix}+\begin{pmatrix}1\\4\\1\end{pmatrix}\right]=\begin{pmatrix}3\\3\\0\end{pmatrix}$ $\pi:\mathbf{r}\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix}=\begin{pmatrix}3\\3\\0\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix}=-6+3=-3$ $\therefore \pi:\mathbf{r}\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix}=-3$
(iv)	$\begin{pmatrix}2\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix}=-4+1=-3$ Hence, plane π contains point P.

(v)	Plane π contains point $P \Rightarrow AP = BP$. Moreover, l_1 and l_2 are not perpendicular. Hence, $\left\lvert \overrightarrow{PA} \times \overrightarrow{PB} \right\rvert$ gives the area of the **rhombus** in which PA and PB are adjacent sides. Do not accept: Area of parallelogram Area of square Area of rectangle $2 \times$area of ΔPAB

Qn	Solution
3	**Complex 3**
(i)	 $\lvert \mathrm{i}z - 3 \rvert < \lvert 3\mathrm{i} \rvert$ $\Rightarrow \lvert \mathrm{i} \rvert \left\lvert z - \frac{3}{\mathrm{i}} \right\rvert < 3$ $\Rightarrow \lvert z - (-3\mathrm{i}) \rvert < 3$
(ii)	Least $\lvert z + 2 + 3\mathrm{i} \rvert = AC$ $= 2\sin\frac{\pi}{4}$ $= \sqrt{2}$ or 1.41 (to 3 s.f.) 
(iii)	Greatest $\arg(z + 2 + 3\mathrm{i}) = \angle DAE$ $= \tan^{-1}\frac{3}{2}$ $= 0.983$ rad (to 3 s.f.) Least $\arg(z + 2 + 3\mathrm{i}) = \angle DAF$ **<u>Method 1</u>** 

since $\angle CAD = \angle CDA = \frac{\pi}{4}$,

$AC = CD = \sqrt{2}$

$\tan\left(\angle FAC\right) = \frac{3-\sqrt{2}}{\sqrt{2}}$

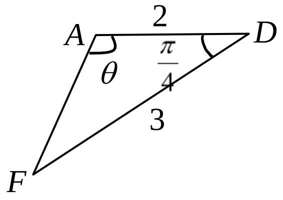
$\angle FAC = 0.84253$ rad

$\therefore$ least $\arg\left(z+2+3\mathrm{i}\right) = -0.84253 - \frac{\pi}{4}$

$= -1.63$ rad (to 3 s.f.)

<u>Method 2</u>

Let $\angle DAF$ be θ, then



$\angle AFD = \pi - \frac{\pi}{4} - \theta$

$= \frac{3\pi}{4} - \theta$

Using sine rule,

$\frac{\sin\angle AFD}{2} = \frac{\sin\angle DAF}{3}$

$\Rightarrow \frac{\sin\left(\frac{3\pi}{4} - \theta\right)}{2} = \frac{\sin\theta}{3}$

$\Rightarrow 2\sin\theta = 3\left[\sin\left(\frac{3\pi}{4}\right)\cos\theta - \cos\left(\frac{3\pi}{4}\right)\sin\theta\right]$

Using GC, $\theta = 1.63$ rad

Hence, least $\arg\left(z+2+3\mathrm{i}\right) = -1.63$ rad

<u>Method 3</u>

From the diagram, co-ordinates of F are

$\left(0 - 3\cos\frac{\pi}{4}, -3 - 3\sin\frac{\pi}{4}\right) = \left(-\frac{3\sqrt{2}}{2}, -3 - \frac{3\sqrt{2}}{2}\right)$

$\angle DAF = \arg\left[\left(-\frac{3\sqrt{2}}{2} + \left(-3 - \frac{3\sqrt{2}}{2}\right)\mathrm{i}\right) - \left(-2 - 3\mathrm{i}\right)\right]$

$= -1.63$ rad

$\therefore -1.63 < \arg(z+2+3\mathrm{i}) < 0.983$ (to 3 s.f.)

Qn	Solution
4	**Application of Differentiation**
(a)	Let the volume of cone be V. By Pythagoras Theorem, $1 = r^2 + (h-1)^2$ $r^2 = 1 - (h^2 - 2h + 1)$ $= 2h - h^2$

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi\left(2h - h^2\right)h$$
$$= \frac{1}{3}\pi\left(2h^2 - h^3\right)$$

At maximum volume $\frac{\mathrm{d}V}{\mathrm{d}h} = 0$

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{1}{3}\pi\left(4h - 3h^2\right) = 0$$
$$4 - 3h = 0$$
$$h = \frac{4}{3}$$

h	$\left(\frac{4}{3}\right)^-$	$\frac{4}{3}$	$\left(\frac{4}{3}\right)^+$
$\frac{\mathrm{d}V}{\mathrm{d}h}$	+	0	–

$\therefore$ V is maximum when $h = \frac{4}{3}$

Sub $h = \frac{4}{3}$ into $V = \frac{1}{3}\pi\left(2h^2 - h^3\right)$

$$V = \frac{1}{3}\pi\left[2\left(\frac{4}{3}\right)^2 - \left(\frac{4}{3}\right)^3\right]$$
$$= \frac{32}{81}\pi$$

(b)(i)	**Method 1** $\cos\theta = \dfrac{h}{\sqrt{r^2+h^2}}$ $\cos\theta = \dfrac{h}{\sqrt{2h-h^2+h^2}}$, using $r^2 = 2h-h^2$ $\cos\theta = \dfrac{h}{\sqrt{2h}}$ $\cos^2\theta = \dfrac{h^2}{2h}$ $\therefore h = 2\cos^2\theta$ **Method 2** Using circle property (i.e. angle at centre = 2 angle at circumference), $\cos 2\theta = \dfrac{h-1}{1}$ $2\cos^2\theta - 1 = h-1$ $\therefore h = 2\cos^2\theta$ (shown) **Method 3** $\cos\theta = \dfrac{\frac{1}{2}\sqrt{h^2+r^2}}{1}$ $= \dfrac{1}{2}\sqrt{h^2+2h-h^2}$ (since $r^2 = 2h-h^2$ when $k=1$) $= \dfrac{1}{2}\sqrt{2h}$ $\therefore \cos^2\theta = \dfrac{h}{2}$ $h = 2\cos^2\theta$ (shown)
(ii)	Method 1: Find V in terms of θ $V = \dfrac{1}{3}\pi\left(2h^2 - h^3\right)$ $= \dfrac{2}{3}\pi k\left(2\cos^2\theta\right)^2 - \dfrac{1}{3}\pi\left(2\cos^2\theta\right)^3$ $= \dfrac{8}{3}\pi\cos^4\theta - \dfrac{8}{3}\pi\cos^6\theta$ $= \dfrac{8}{3}\pi\left(\cos^4\theta - \cos^6\theta\right)$ $\dfrac{\mathrm{d}V}{\mathrm{d}\theta} = \dfrac{8}{3}\pi\left[4\cos^3\theta(-\sin\theta) - 6\cos^5\theta(-\sin\theta)\right]$ $= \dfrac{16}{3}\pi\cos^3\theta\sin\theta\left(3\cos^2\theta - 2\right)$

When $h=\frac{3}{2}$,

$$\frac{3}{2}=2\cos^2\theta$$

$$\cos^2\theta=\frac{3}{4}$$

$$\cos\theta=\pm\frac{\sqrt{3}}{2}$$

Since θ is acute,

$$\theta=\frac{\pi}{6}$$

Therefore $\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{\mathrm{d}\theta}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$

$$=\frac{1}{\frac{16}{3}\pi\cos^3\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}\right)\left(3\cos^2\left(\frac{\pi}{6}\right)-2\right)}\times 6$$

$$=\frac{1}{\frac{16}{3}\pi\left(\frac{\sqrt{3}}{2}\right)^3\left(\frac{1}{2}\right)\left[3\left(\frac{\sqrt{3}}{2}\right)^2-2\right]}\times 6$$

$$=\frac{4}{\pi\sqrt{3}}\times 6$$

$$=\frac{24}{\pi\sqrt{3}}$$

$$=\frac{8\sqrt{3}}{\pi}\text{ rad/s}$$

Method 2 (**recommended**):

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{\mathrm{d}\theta}{\mathrm{d}h}\times\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$$

$$h=2\cos^2\theta$$

$$\frac{\mathrm{d}h}{\mathrm{d}\theta}=4\cos\theta(-\sin\theta)$$

$$=-2\sin 2\theta$$

$$\therefore\frac{\mathrm{d}\theta}{\mathrm{d}h}=\frac{1}{-2\sin 2\theta}$$

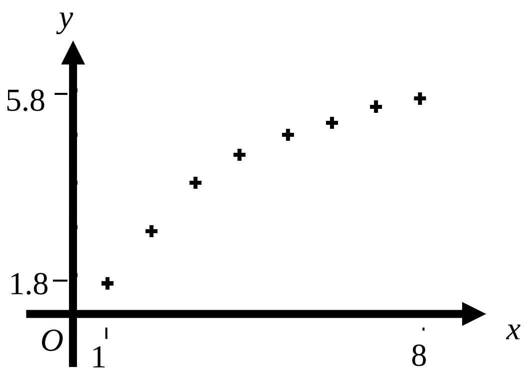
From (a) $\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{1}{3}\pi\left(4h-3h^2\right)$

When $h=\frac{3}{2}$,

	$\frac{3}{2}=2\cos^2\theta$ $\cos^2\theta=\frac{3}{4}$ $\cos\theta=\pm\frac{\sqrt{3}}{2}$ Since θ is acute, $\theta=\frac{\pi}{6}$ $\therefore \frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{\mathrm{d}\theta}{\mathrm{d}h}\times\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$ $\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{1}{-2\sin 2\left(\frac{\pi}{6}\right)}\times\frac{3}{\pi\left(4\left(\frac{3}{2}\right)-3\left(\frac{3}{2}\right)^2\right)}\times 6$ $=\frac{1}{-\sqrt{3}}\times\frac{4}{-\pi}\times 6$ $=\frac{24}{\pi\sqrt{3}}$ $=\frac{8\sqrt{3}}{\pi}$ rad/s Therefore θ is increasing at a rate of $\frac{8\sqrt{3}}{\pi}$ rad/s

Qn	Solution
5	**Probability**
(i)	$\mathrm{P}(\text{no number matched})=\frac{\binom{38}{8}}{\binom{45}{8}}$ $=0.226874$ $=0.227$ (to 3 s.f.)
(ii)	$\mathrm{P}(\text{4 numbers matched} \mid \text{at least 1 number matched})=\frac{\mathrm{P}(\text{4 numbers matched})}{1-\mathrm{P}(\text{no number matched})}$ $=\frac{\frac{\binom{7}{4}\binom{38}{4}}{\binom{45}{8}}}{1-0.227}$ $=0.015502$ $=0.0155$ (to 3 s.f.)

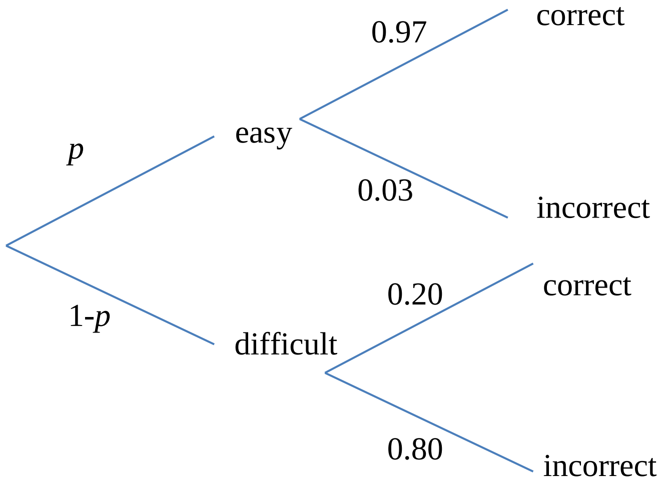
Qn	Solution
6	**Sampling Methods**
(i)	Obtain a list of all the students and number them from 1 to 1800. Determine the sampling interval = $\frac{1800}{180}=10$. Choose a **random** start by selecting a **random** number from 1 to 10. e.g. 5 Finally select every 10th student thereafter (e.g. 5th , 15th , 25th,…) until 180 students are obtained.
(ii)	Stratified Sampling. Choose **random samples** of the following sizes from each group: 49 Year 1 boys, 38 Year 1 girls, 56 Year 2 boys and 37 Year 2 girls .
(iii)	It will not be a random sample as not every student in the college has an equal probability (or chance) of being selected, for example those who are absent will not be selected.

Qn	Solution
7	**Correlation & Regression**
(i)	
(ii)	Using G.C., $r=0.952$ (to 3 s.f.) The linear model may not be suitable because the scatter diagram shows that as *x* increases, *y* increases at a decreasing rate. OR In the context of the question, it is not possible for the height of a tree to increase indefinitely, hence a linear model is not suitable.
(iii)	A quadratic model is not suitable for long-term predictions, as a quadratic model with a maximum point means that there will be a point in time where, as the value of *x* increases, *y* decreases. But *y* (height) can only increase or maintain the same as years go by. Hence, this is not an appropriate model.
(iv)	Using G.C., regression line is $y=1.9888h+1.7388 \Rightarrow y=1.99h+1.74$. $r=0.997$

(v)	When $y=5.2$ and using regression line $y=1.9888(\ln x)+1.7388$, $\Rightarrow 5.2=1.9888(\ln x)+1.7388$ $\Rightarrow \ln x=1.7403$ $\Rightarrow x=5.70$ years

Qn	Solution
8	**Hypothesis Testing**
(i)	Let $U=X-50$ Unbiased estimate of μ is $\bar{x}=\dfrac{\sum(x-50)}{n}+50=\dfrac{3916}{80}+50=98.95$ Unbiased estimate of σ^2 is $s_x{}^2=s_u{}^2=\dfrac{1}{n-1}\left[\sum u^2-\dfrac{\left(\sum u\right)^2}{n}\right]$ $=\dfrac{1}{79}\left[193375-\dfrac{(3916)^2}{80}\right]$ $=21.352$ $=21.4$(3 s.f)
(ii)	Let X be the mass of the contents of a randomly chosen packet of nuts, in grams. Let μ denote the population mean mass of the contents of packets of nuts, in grams. H_0: $\mu=100$ H_1: $\mu<100$ Since $n=80$ is large, by the Central Limit Theorem, $\bar{X}\sim N\left(\mu,\dfrac{\sigma^2}{n}\right)$ approximately. Test statistic: $Z=\dfrac{\bar{X}-\mu}{S/\sqrt{n}}$ Level of significance: 4% Reject H_0 if p-value < 0.04 Under H_0, using G.C., p-value = 0.021055 Conclusion: Since p-value = 0.0211 < 0.04, we reject H_0 and conclude that there is sufficient evidence, at 4% level, that the population mean mass of the contents is overstated.
(iii)	"4% significance level" means there is a probability of 0.04 of concluding that the population mean mass of contents in packets of nuts is less than 100 grams when, in fact, it is 100 grams.
(iv)	If we test $H_0:\mu=100$ $H_1:\mu\neq 100$

	New p-value $= 2 \times \text{old } p\text{-value}$ $= 2(0.021055)$ $= 0.04211 > 0.04$ Hence, the conclusion would not be the same.

Qn	**Solution**
9	**Probability**
	
(i)	Required probability $= (0.97p)^3(1-p)(0.80) \times \frac{4!}{3!} = 0.18$ Using GC, $p = 0.497$ or $p = 0.92115$ Since he has the tendency to choose easy question over difficult question, $p = 0.921$
(ii)	Required probability $= (0.97 \times 0.921)^2 (0.03 \times 0.921)^2 \times \frac{4!}{2!2!}$ $= 0.00366$ (3 s.f.)
(iii)	To score 4 points and at most 1 difficult question, Cases are 4 easy correct $= (0.97 \times 0.921)^4 = 0.636978$ 2 easy correct , 1 easy incorrect, 1 difficult correct $= (0.921 \times 0.97)^2 \left[(0.921)(0.03)\right](1-0.921)0.20 \times \frac{4!}{2!} = 0.0041810$ Required probability $= \dfrac{0.63698 + 0.0041810}{0.921^4 + 0.921^3(1-0.921)\left(\frac{4!}{3!}\right)}$ $= 0.66346$ $= 0.663$ **(to 3 s.f.)**

Qn	Solution
10	**Normal Distribution**
(i)	Let X and Y denote the mass, in kilograms, for a randomly chosen pumpkin and a randomly chosen watermelon respectively. $X \sim \mathrm{N}\left(4.2, 1.6^2\right)$ $\quad Y \sim \mathrm{N}\left(8.5, 2.2^2\right)$ Let $\bar{Y} = \dfrac{Y_1 + Y_2 + \cdots + Y_5}{5}$. $\bar{Y} \sim \mathrm{N}\left(8.5, \dfrac{2.2^2}{5}\right)$ $\mathrm{P}\left(\bar{Y} > k\right) \le 0.10$ Using GC, $k \ge 9.76$ (to 3 s.f.)
(ii)	$W = X_1 + X_2 + \cdots + X_5 \sim \mathrm{N}\left(4.2(5), 1.6^2(5)\right)$ $W \sim \mathrm{N}\left(21, 12.8\right)$ $\mathrm{E}(W - 3\bar{Y}) = 21 - 3 \times 8.5 = -4.5$ $\mathrm{Var}(W - 3\bar{Y}) = 12.8 + 3^2 \times \dfrac{2.2^2}{5} = 21.512$ $W - 3\bar{Y} \sim N(-4.5, 21.512)$ $\mathrm{P}\left(\left\lvert W - 3\bar{Y} \right\rvert > 5\right) = \mathrm{P}\left(W - 3\bar{Y} > 5\right) + \mathrm{P}\left(W - 3\bar{Y} < -5\right)$ $= 0.477$ (to 3 s.f.)
(iii)	Let S be the number of rotten pumpkins out of the n pumpkins in a batch. $S \sim B\left(n, \dfrac{3}{20}\right)$ $\mathrm{E}\left(S\right) = n \times \dfrac{3}{20} = \dfrac{3n}{20}$ $\quad \mathrm{Var}\left(S\right) = n \times \dfrac{3}{20} \times \left(1 - \dfrac{3}{20}\right) = \dfrac{51n}{400}$ Since n = 70 is large, by Central Limit Theorem, $\bar{S} \sim N\left(\dfrac{3}{20}n, \dfrac{\frac{51n}{400}}{70}\right)$ approximately. Given $P(\bar{S} \ge 4) > 0.7$ Using GC, When n = 27, $P(\bar{S} \ge 4)$ = 0.589 < 0.7 When n = 28, $P(\bar{S} \ge 4)$ = 0.812 > 0.7 Hence minimum value of n is 28.

Qn	Solution
11	**Binomial and Poisson**
(a)(i)	Let X be number of people convicted of robbery in a month. $$X \sim \text{Po}(2\times 4)$$ $$\Rightarrow X \sim \text{Po}(8)$$ Required probability $=\text{P}(X \geq 9)$ $=1-\text{P}(X \leq 8)$ $=0.40745$ $=0.407 \quad (3\text{sf})$
(ii)	Let Y be number of people convicted of robbery in a year of 52 weeks. $$Y \sim \text{Po}(2\text{x}52)$$ $$Y \sim \text{Po}(104)$$ Since $\lambda = 104 > 10$, $\quad Y \sim N(104,104)$ approximately Required probability $=\text{P}(Y \leq 90)$ $=\text{P}(Y < 90.5)$ after continuity correction $=0.092787$ $=0.0928 \quad (3\text{sf})$
(iii)	The MEAN number of people convicted would not be constant every week because during holiday seasons/economic downturn/recession/tourist peak periods, there could be an increase in robbery cases.
(b)(i)	The probability that it rains is the same for each day in June.
(ii)	Let X be the number of days, out of 15, that rains. $X \sim \text{B}(15, p)$ $$\text{P}(X \geq 1) = 0.96482$$ $$1-\text{P}(X=0) = 0.96482$$ $$\text{P}(X=0) = 0.03518$$ $$\binom{15}{0} p^0 (1-p)^{15} = 0.03518$$ $$(1-p)^{15} = 0.03518$$ $$p = 0.2$$
(iii)	Let Y be the number of days, out of 30, that rains. $Y \sim \text{B}(30, 0.25)$ From G.C., When n = 6, $\text{P}(Y=6) = 0.14546$ When n = 7, $\text{P}(Y=7) = 0.16624$ When n = 8, $\text{P}(Y=8) = 0.15931$ Therefore, the most likely number of rainy days is 7.