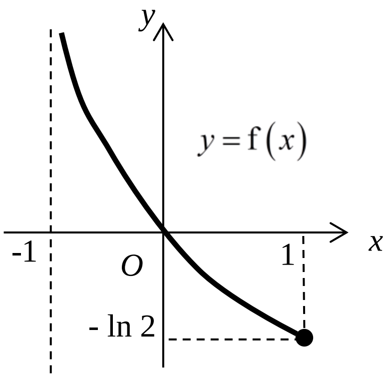
| Qn | Solution |
|---|---|
| **1** | **Mathematical Induction** |
| | Let $P_n$ be the statement $\sum_{r=1}^{n}\frac{r-1}{3^r}=\frac{1}{4}-\frac{1}{4}\left(\frac{1}{3}\right)^n(2n+1)$ for $n\in\mathbb{Z}^+$. <br> When $n=1$ <br> $\text{LHS}=\sum_{r=1}^{1}\frac{r-1}{3^r}=\frac{1-1}{3^1}=0$ <br> $\text{RHS}=\frac{1}{4}-\frac{1}{4}\left(\frac{1}{3}\right)^0(0+1)=0=\text{LHS}$ <br> $\therefore P_1$ is true. <br><br> Assume $P_k$ is true for some $k\in\mathbb{Z}^+$, i.e. $\sum_{r=1}^{k}\frac{r-1}{3^r}=\frac{1}{4}-\frac{1}{4}\left(\frac{1}{3}\right)^k(2k+1)$ <br> To show that $P_{k+1}$ is true, i.e. $\sum_{r=1}^{k+1}\frac{r-1}{3^r}=\frac{1}{4}-\frac{1}{4}\left(\frac{1}{3}\right)^{k+1}(2k+3)$ <br><br> $\text{LHS}=\sum_{r=1}^{k+1}\frac{r-1}{3^r}=\sum_{r=1}^{k}\frac{r-1}{3^r}+\frac{k}{3^{k+1}}$ <br> $=\frac{1}{4}-\frac{1}{4}\left(\frac{1}{3}\right)^k(2k+1)+\frac{k}{3^{k+1}}$ <br> $=\frac{1}{4}-\frac{1}{4}\left(\frac{1}{3}\right)^{k+1}\left(3(2k+1)-4k\right)$ <br> $=\frac{1}{4}-\frac{1}{4}\left(\frac{1}{3}\right)^{k+1}(6k+3-4k)$ <br> $=\frac{1}{4}-\frac{1}{4}\left(\frac{1}{3}\right)^{k+1}(2k+3)=\text{RHS}$ <br><br> Thus $P_k$ is true $\Rightarrow$ $P_{k+1}$ is true. <br><br> Since $P_1$ is true and $P_k$ is true $\Rightarrow$ $P_{k+1}$ is true, by mathematical induction, $P_n$ is true for $n\in\mathbb{Z}^+$. |

| Qn | Solution |
|---|---|
| **2** | **Functions** |

**(i)**




Since any horizontal line $y = k, k \in \mathbb{R}$ cuts the graph of $y = \mathrm{f}(x)$ at most once, f is a one-one function and hence $\mathrm{f}^{-1}$ exists. OR

From the graph above, $y = \mathrm{f}(x)$ is strictly decreasing on its domain $-1 < x \leq 1$, f is a one-one function and hence $\mathrm{f}^{-1}$ exists.

If no graph:

$\mathrm{f}'(x) = -\dfrac{1}{1+x} < 0$ for all $-1 < x \leq 1$, hence $y = \mathrm{f}(x)$ is strictly decreasing on its domain $-1 < x \leq 1$, f is a one-one function and hence $\mathrm{f}^{-1}$ exists.

**(ii)**

Let $y = \ln\left(\dfrac{1}{1+x}\right)$

$\dfrac{1}{1+x} = \mathrm{e}^{y}$

$x = \mathrm{e}^{-y} - 1$

$\therefore \mathrm{f}^{-1}(x) = \mathrm{e}^{-x} - 1$

$\mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R}_{\mathrm{f}} = [-\ln 2, \infty)$

$\therefore \mathrm{f}^{-1} : x \mapsto \mathrm{e}^{-x} - 1,\ x \in \mathbb{R},\ x \geq -\ln 2$

**(iii)**

(a)

For $\mathrm{fg}(x) = x$ to be true, it implies that $\mathrm{g}(x) = \mathrm{f}^{-1}(x)$,

Therefore $\mathrm{g}(x) = \mathrm{e}^{-x} - 1$

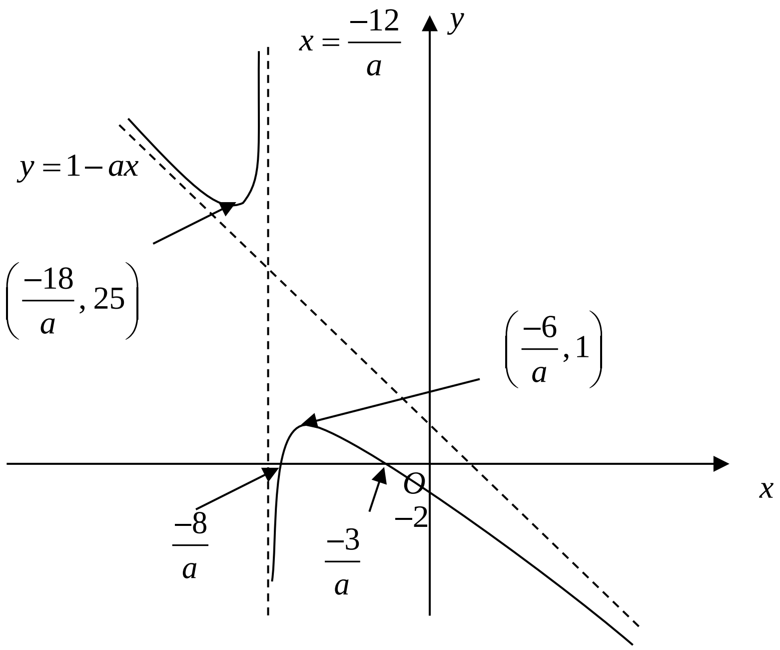
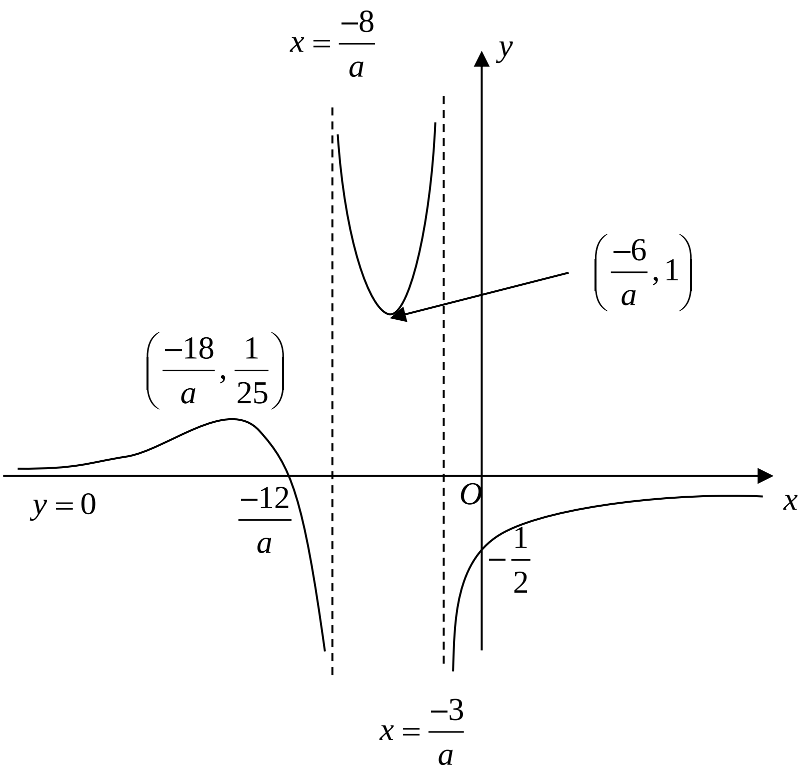
(b)

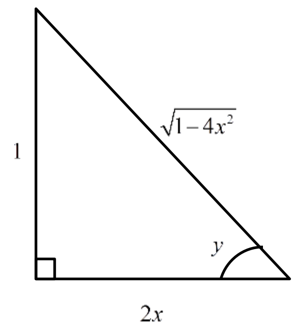
$\mathrm{gf}(x) = (x+1)^2$

$\Rightarrow \mathrm{gf}\left(\mathrm{f}^{-1}(x)\right) = \left(\mathrm{f}^{-1}(x) + 1\right)^2$

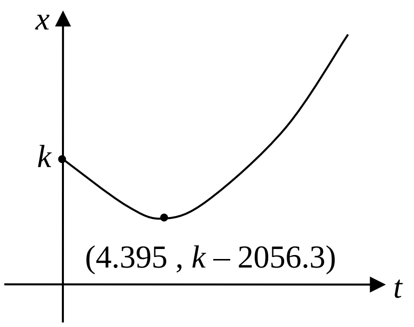
$\Rightarrow \mathrm{g}(x) = \left(\left(\mathrm{e}^{-x} - 1\right) + 1\right)^2 = \mathrm{e}^{-2x}$

| Qn | Solution |
|---|---|
| **3** | **Recurrence Relations** |
| **(i)** | $x-\ln(x+2)=0$<br>Using GC (graph), $\alpha=-1.8414$ and $\beta=1.1462$ |
| **(ii)** | As $n\to\infty, x_n\to L, x_{n+1}\to L$<br>$L=\ln(L+2)$<br>$\Rightarrow \mathrm{e}^L=L+2$<br>$\Rightarrow L+2-\mathrm{e}^L=0$<br>$\therefore L=\alpha$ or $L=\beta$<br>Since the sequence is positive, $L\geq 0$, $L=\beta$.<br>Therefore the sequence converges to $\beta$.<br><br>Note: A positive sequence can converge to zero it is not correct to say $L > 0$. |
| **(iii)** | For $x_1=2$, the sequence **decreases** and **converges to** $\beta=1.1462$**.** |
| **(iv)** | When $x_n>\beta$, from the graph,<br>$x_n+2-\mathrm{e}^{x_n}<0$<br>$\Rightarrow x_n+2<\mathrm{e}^{x_n}$<br>$\Rightarrow \ln(x_n+2)<x_n$<br>$\Rightarrow x_{n+1}<x_n$. |

| Qn | Solution |
|---|---|
| 4 | **Graphing Techniques** |
| (i) |  |
| (ii) |  |
| | Least value of $k=a$ |

| Qn | Solution |
|---|---|
| **5** | **Maclaurin Series & Binomial Theorem** |
| | $y=\cos^{-1}(2x)$<br>$\cos y = 2x$<br>Differentiate wrt *x*,<br>$-\sin y\dfrac{\mathrm{d}y}{\mathrm{d}x}=2$<br>Differentiate wrt *x*,<br>$-\sin y\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\dfrac{\mathrm{d}y}{\mathrm{d}x}\left(-\cos y\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)=0$<br>$\sin y\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$ (proven)<br><br>Alternative Method (not recommended)<br>$y=\cos^{-1}(2x)$<br>Differentiate wrt *x*,<br>$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-2}{\sqrt{1-4x^2}}$<br>$\sqrt{1-4x^2}\dfrac{\mathrm{d}y}{\mathrm{d}x}=-2$<br>Differentiate wrt *x*,<br>$\sqrt{1-4x^2}\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\dfrac{\mathrm{d}y}{\mathrm{d}x}\dfrac{-8x}{2\sqrt{1-4x^2}}=0$<br>$\sin y\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\dfrac{\mathrm{d}y}{\mathrm{d}x}\dfrac{-2}{\sqrt{1-4x^2}}(2x)=0$<br>$\sin y\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$ (proven)<br><br>Using $\cos y = 2x$<br>$\sin y=\sqrt{1-4x^2}$ |
| **(i)** | $\sin y\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$<br>Differentiate wrt *x*,<br>$\sin y\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+\cos y\dfrac{\mathrm{d}y}{\mathrm{d}x}\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+4x\dfrac{\mathrm{d}y}{\mathrm{d}x}\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$<br>$\sin y\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+(\cos y+4x)\dfrac{\mathrm{d}y}{\mathrm{d}x}\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$<br><br>When $x=0$, |

| | |
|---|---|
| | $y=\cos^{-1}(0)=\dfrac{\pi}{2}$<br>$-\sin\left(\dfrac{\pi}{2}\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}=2\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2}{-1}=-2$<br>$\sin\left(\dfrac{\pi}{2}\right)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2(0)(-2)^2=0\Rightarrow\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$<br>$\sin\left(\dfrac{\pi}{2}\right)\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+\left(\cos\dfrac{\pi}{2}+4(0)\right)(-2)(0)+2(-2)^2\Rightarrow\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=-8$<br>$\therefore y=\dfrac{\pi}{2}-2x-\dfrac{4}{3}x^3+\cdots$ |
| **(ii)** | $\dfrac{\cos^{-1}(2x)}{\sqrt{1-3x^2}}=\left(\dfrac{\pi}{2}-2x+\cdots\right)\left(1-3x^2\right)^{-\frac{1}{2}}$<br>$=\left(\dfrac{\pi}{2}-2x+\cdots\right)\left(1+\dfrac{3}{2}x^2+...\right)$<br>$=\dfrac{\pi}{2}-2x+\dfrac{3\pi}{4}x^2+...$ |

| Qn | Solution |
|---|---|
| 6 | **Differential Equations** |
| (i) | $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{x-k}{t}+1000-\frac{5000}{t+5}$ |
| (ii) | $y=\frac{x-k}{t}$<br>$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{t\frac{\mathrm{d}x}{\mathrm{d}t}-(x-k)}{t^2}$<br>$t^2\frac{\mathrm{d}y}{\mathrm{d}t}=t\left(\frac{x-k}{t}+1000-\frac{5000}{t+5}\right)-(x-k)$<br>$t^2\frac{\mathrm{d}y}{\mathrm{d}t}=1000t-\frac{5000t}{t+5}$<br>$t^2\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{1000t(t+5)-5000t}{t+5}$<br>$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{1000}{t+5}$ (shown) |
| (iii) | Integrating both sides wrt $t$,<br>$y=\int\frac{1000}{t+5}\,\mathrm{d}t=1000\ln\lvert t+5\rvert+C$<br>$\frac{x-k}{t}=1000\ln\lvert t+5\rvert+C$<br>$x=k+1000t\ln\lvert t+5\rvert+Ct$<br>Since $t\geq 0$, $x=k+1000t\ln(t+5)+Ct$ |
| | When $t=10$, $x=k$<br>$\frac{k-k}{t}=1000\ln(10+5)+C\Rightarrow 0=1000\ln 15+C\Rightarrow C=-1000\ln 15$<br>$x=k+1000t\ln(t+5)-(1000\ln 15)t\Rightarrow x=k+1000t\ln\left(\frac{t+5}{15}\right)$<br>Consider the graph of $x=k+1000t\ln\left(\frac{t+5}{15}\right)$ for $t\geq 0$.<br> <br>In order for the trees not to become extinct,<br>$x>0$<br>$k-2056.3>0$<br>$k>2056.3$<br>The minimum initial number of trees is 2057. |

| Qn | Solution |
|---|---|
| **7** | **Vectors + System of Linear Equations** |
| **(i)** | $p_1 : \mathbf{r} \cdot \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix} = 13$<br><br>$p_2 : \mathbf{r} \cdot \begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix} = -1$<br><br>Let the acute angle between $p_1$ and $p_2$ be $\theta$.<br><br>$\cos\theta = \dfrac{\left\lvert \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix} \right\rvert}{\sqrt{30}\sqrt{30}}$<br><br>$\theta = \cos^{-1}\left(\dfrac{3}{30}\right)$<br><br>$\theta = 84.261°$<br><br>$\therefore \theta = 84.3°$ |
| **(ii)** | $p_1 : x - 5y + 2z = 13$<br>$p_2 : 2x - y - 5z = -1$<br><br>Let $z = \alpha$, $\alpha \in \mathbb{R}$ , $x = -2 + 3\alpha$<br>$y = -3 + \alpha$<br><br>$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2+\alpha \\ -3+\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$<br><br>Vector equation of line $l : \mathbf{r} = \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$, $\alpha \in \mathbb{R}$ |
| **(iii)** | For $p_1$, $p_2$ and $p_3$ to meet in the line $l$ , $l$ lies in $p_3$.<br><br>Since $l$ lies in plane $p_3$, the direction vector of $l$ is perpendicular to the normal of $p_3$ and any point on line $l$ will satisfy the equation of the plane $p_3$. Thus<br><br>$\begin{pmatrix} \lambda \\ -4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = 0 \Rightarrow 3\lambda - 4 + 1 = 0 \Rightarrow \lambda = 1$<br><br>and $(-2, -3, 0)$ satisfies equation of $p_3$ i.e. $\begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} = \mu$<br><br>$(-2)(1) - (-3)(-4) = \mu \Rightarrow \mu = 10$ |

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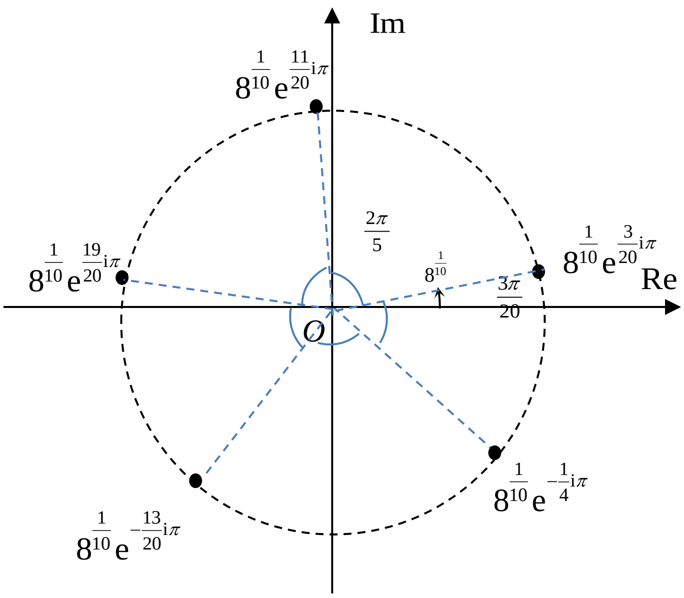
| | $\therefore\ \lambda = 1$<br>$\mu = 10$ |
| **(iv)** | The three planes $p_1$, $p_2$ and $p_3$ form a triangular prism and do not have a common point of intersection.<br><br>$l$ is still perpendicular to the normal of $p_3$<br>$\therefore\ \lambda = 1$<br><br>but $(-2,-3,0)$ does not satisfies equation of $p_3$ $\therefore$ $\mu \neq 10$ |

| **Qn** | **Solution** |
|---|---|
| **8** | **Complex Numbers 2** |
| **(a)** | $z^5 - \left(-2+2\mathrm{i}\right) = 0$<br>$z^5 = -2+2\mathrm{i}$<br>$z^5 = \sqrt{8}\mathrm{e}^{\frac{3}{4}\pi\mathrm{i}}$<br>$z^5 = \sqrt{8}\mathrm{e}^{\mathrm{i}\pi\left(\frac{3}{4}+2k\right)}$<br>$z = 8^{\frac{1}{10}}\mathrm{e}^{\mathrm{i}\frac{\pi}{5}\left(\frac{3}{4}+2k\right)}$ where $k = -2,-1,0,1,2$<br>$= 8^{\frac{1}{10}}\mathrm{e}^{-\frac{13}{20}\mathrm{i}\pi},\ 8^{\frac{1}{10}}\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi},\ 8^{\frac{1}{10}}\mathrm{e}^{\frac{3}{20}\mathrm{i}\pi},\ 8^{\frac{1}{10}}\mathrm{e}^{\frac{11}{20}\mathrm{i}\pi},\ 8^{\frac{1}{10}}\mathrm{e}^{\frac{19}{20}\mathrm{i}\pi}$<br><br>Hence the roots are $8^{\frac{1}{10}}\mathrm{e}^{-\frac{13}{20}\mathrm{i}\pi},\ 8^{\frac{1}{10}}\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi},\ 8^{\frac{1}{10}}\mathrm{e}^{\frac{3}{20}\mathrm{i}\pi},\ 8^{\frac{1}{10}}\mathrm{e}^{\frac{11}{20}\mathrm{i}\pi},\ 8^{\frac{1}{10}}\mathrm{e}^{\frac{19}{20}\mathrm{i}\pi}$. |
| |  |
| **(b)** | $\mathrm{e}^{\mathrm{i}\frac{\theta}{2}} - \mathrm{e}^{-\mathrm{i}\frac{\theta}{2}} = \left(\cos\frac{\theta}{2} + \mathrm{i}\sin\frac{\theta}{2}\right) - \left(\cos\frac{\theta}{2} - \mathrm{i}\sin\frac{\theta}{2}\right) = 2\mathrm{i}\sin\frac{\theta}{2}$ (shown) |

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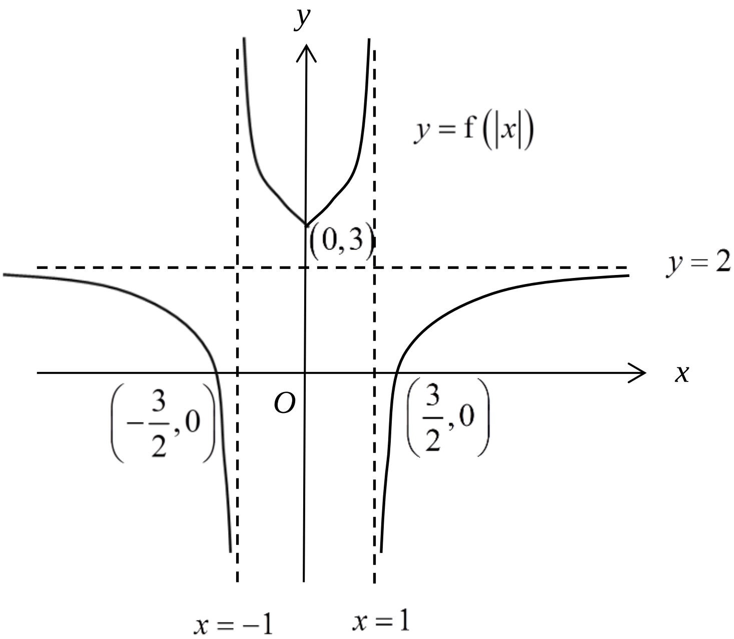
|---|---|
| | $$\frac{e^{i\theta}}{1-e^{i\theta}}=\frac{e^{i\theta}}{e^{i\frac{\theta}{2}}\left(e^{-i\frac{\theta}{2}}-e^{i\frac{\theta}{2}}\right)}$$ $$=\frac{e^{i\theta}e^{-i\frac{\theta}{2}}}{-\left(-e^{-i\frac{\theta}{2}}+e^{i\frac{\theta}{2}}\right)}$$ $$=\frac{e^{i\frac{\theta}{2}}}{-2i\sin\frac{\theta}{2}}$$ $$=-\frac{1}{2i}\frac{\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}}$$ $$=\frac{i}{2}\left(\cot\frac{\theta}{2}+i\right)$$ $$=\frac{1}{2}\left(i\cot\frac{\theta}{2}-1\right) \qquad \text{(shown)}$$ |

| Qn | Solution |
|---|---|
| 9 | **Definite Integral** |
| (i) |   |

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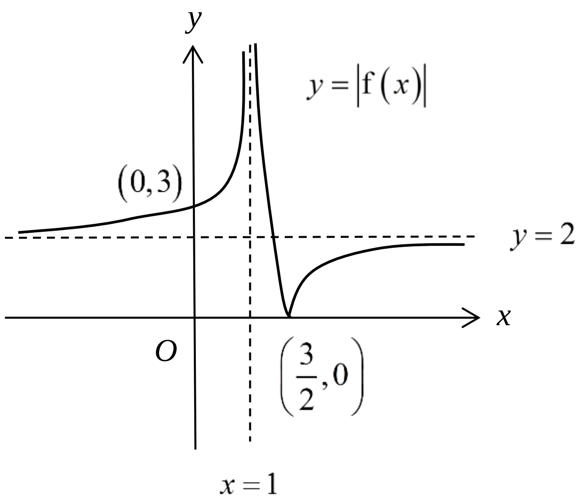
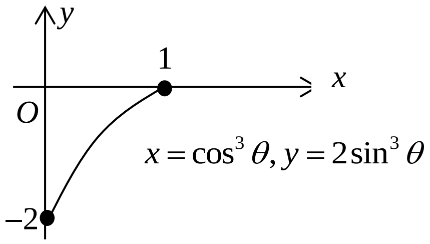
| |  |
| **(ii)** | Required area $=\int_{\frac{5}{4}}^{2} \mathrm{f}\left(\lvert x\rvert\right)\ \mathrm{d}x$<br>$=-\int_{\frac{5}{4}}^{\frac{3}{2}}\left(2-\frac{1}{x-1}\right)\ \mathrm{d}x+\int_{\frac{3}{2}}^{2}\left(2-\frac{1}{x-1}\right)\ \mathrm{d}x$<br>$=-\left[2x-\ln\lvert x-1\rvert\right]_{\frac{5}{4}}^{\frac{3}{2}}+\left[2x-\ln\lvert x-1\rvert\right]_{\frac{3}{2}}^{2}$<br>$=-\left[\left(3-\ln\frac{1}{2}\right)-\left(\frac{5}{2}-\ln\frac{1}{4}\right)\right]+\left[\left(4-\ln 1\right)-\left(3-\ln\frac{1}{2}\right)\right]$<br>$=-\left(\frac{1}{2}-\ln 2\right)+\left[1-\ln 2\right]$<br>$=-\frac{1}{2}+\ln 2+\left[1-\ln 2\right]=\frac{1}{2}$ |
| **(iii)** | Need to find the equation of the reflected portion of the graph:<br>$y=-\left(2-\frac{1}{x-1}\right)$<br>$\frac{1}{x-1}=2+y$<br>$x=\frac{1}{2+y}+1$<br>Required vol $=\pi\left(\frac{3}{2}\right)^2(2)-\pi\int_0^2\left(\frac{1}{2+y}+1\right)^2\ \mathrm{d}y$<br>$=2.71$ |

| Qn | Solution |
|---|---|
| **10** | **Tangent and Normal – Parametric Equations** |
| **(i)** |  |
| **(ii)** | $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6\sin^2\theta\cos\theta}{-3\cos^2\theta\sin\theta} = -2\tan\theta$ |
| **(iii)** | Equation of tangent at $P\left(\cos^3 t,\ 2\sin^3 t\right)$ is<br>$y - 2\sin^3 t = -2\tan t(x - \cos^3 t)$<br><br>At $x$ =0,<br>$y = -2\tan t(-\cos^3 t) + 2\sin^3 t$<br>$= 2\sin t\cos^2 t + 2\sin^3 t$<br>$= 2\sin t\left(1 - \sin^2 t\right) + 2\sin^3 t$<br>$= 2\sin t$<br><br>At $y$ =0,<br>$-2\sin^3 t = -2\tan t(x - \cos^3 t)$<br>$x\left(2\tan t\right) = 2\tan t\cos^3 t + 2\sin^3 t$<br>$x\left(2\tan t\right) = 2\sin t$<br>$x = \frac{2\sin t}{2\tan t} = \cos t$<br><br>Therefore $a$ = 1<br><br>Coordinates of $U$ and $V$ are $(\cos t, 0)$ and $(0, 2\sin t)$ |
| **(iv)** | Midpoint of UV is $\left(\frac{\cos t}{2}, \frac{2\sin t}{2}\right) = \left(\frac{\cos t}{2}, \sin t\right)$<br><br>$x = \frac{\cos t}{2},\ \ y = \sin t$<br>$2x = \cos t,\ \ y = \sin t$<br>$\cos^2 t + \sin^2 t = 1$<br>$4x^2 + y^2 = 1$<br>Therefore, cartesian equation of the locus of the mid-point of $UV$ as $t$ varies is $4x^2 + y^2 = 1$ where $-1 \le y \le 0$, $0 \le x \le 0.5$. |

| Qn | Solution |
|---|---|
| **11** | **Summation & Integration** |
| **ai** | LHS= $(k+1)!k-k!(k-1)$<br>$=k!\left(k(k+1)-(k-1)\right)$<br>$=k!(k^2+1)$ = RHS |
| **(ii)** | $\sum_{k=1}^{n}k!(k^2+1)=\sum_{k=1}^{n}\left[(k+1)!k-k!(k-1)\right]$<br>$=\cancel{2!(1)}-1!(0)$<br>$+\cancel{3!(2)}-\cancel{2!(1)}$<br>$+4!(3)-\cancel{3!(2)}$<br>$+\ldots$<br>$+\cancel{(n-1)!(n-2)}-\cancel{(n-2)!(n-3)}$<br>$+\cancel{n!(n-1)}-\cancel{(n-1)!(n-2)}$<br>$+(n+1)!n-\cancel{n!(n-1)}$<br>$=(n+1)!n$ |
| **(iii)** | $\sum_{k=1}^{n-1}(k+1)!(k^2+2k+2)=\sum_{k-1=1}^{k-1=n-1}(k-1+1)!((k-1)^2+2(k-1)+2)$ (replace $k$ by $k-1$)<br>$=\sum_{k=2}^{n}k!(k^2+1)$<br>$=\sum_{k=1}^{n}k!(k^2+1)-1!(1^2+1)$<br>$=(n+1)!n-2$<br><br>Alternative Method: (Strongly not recommended)<br>Result: $\sum_{k=1}^{n}k!(k^2+1)=(n+1)!n$<br>Replace $k$ by $k+1$,<br>$\sum_{k+1=1}^{k+1=n}(k+1)!((k+1)^2+1)=(n+1)!n$<br>$\Rightarrow\sum_{k=0}^{n-1}(k+1)!(k^2+2k+2)=(n+1)!n$<br>$\therefore\sum_{k=1}^{n-1}(k+1)!(k^2+2k+2)$<br>$=\sum_{k=0}^{n-1}(k+1)!(k^2+2k+2)-(0+1)!(0^2+2(0)+2)$<br>$=\sum_{k=1}^{n}k!(k^2+1)-2$<br>$=(n+1)!n-2$ |

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| **(b) (i)** | $A=\frac{1}{n}\left[\mathrm{e}^{\frac{2}{n}+1}+\mathrm{e}^{\frac{4}{n}+1}+\ldots+\mathrm{e}^{\frac{2(n-1)}{n}+1}+\mathrm{e}^{3}\right]$ <br> $=\frac{\mathrm{e}}{n}\left[\mathrm{e}^{\frac{2}{n}}+\mathrm{e}^{\frac{4}{n}}+\ldots+\mathrm{e}^{\frac{2(n-1)}{n}}+\mathrm{e}^{2}\right]$ <br> $=\frac{\mathrm{e}}{n}\left[\frac{\mathrm{e}^{\frac{2}{n}}\left(1-\left(\mathrm{e}^{\frac{2}{n}}\right)^{n}\right)}{1-\mathrm{e}^{\frac{2}{n}}}\right]$ (Applying GP sum formula) <br> $=\frac{\mathrm{e}}{n}\left[\frac{\mathrm{e}^{\frac{2}{n}}\left(1-\mathrm{e}^{2}\right)}{1-\mathrm{e}^{\frac{2}{n}}}\right]$ (shown) |
| **(ii)** | As $n\to\infty$, <br> $A\to\int_0^1 \mathrm{e}^{2x+1}\,\mathrm{d}x=\left[\frac{1}{2}\mathrm{e}^{2x+1}\right]_0^1$ <br> $=\frac{1}{2}\mathrm{e}^{3}-\frac{1}{2}\mathrm{e}$ <br> $=\frac{\mathrm{e}}{2}\left(\mathrm{e}^{2}-1\right)$ |
| **(iii)** | Since the sum of the area of rectangles is an overestimate, <br> $\frac{\mathrm{e}}{n}\left[\frac{\mathrm{e}^{\frac{2}{n}}\left(1-\mathrm{e}^{2}\right)}{1-\mathrm{e}^{\frac{2}{n}}}\right]>\frac{\mathrm{e}}{2}\left(\mathrm{e}^{2}-1\right)$ <br> $\Rightarrow\frac{\mathrm{e}}{n}\left[\frac{\mathrm{e}^{\frac{2}{n}}\left(\mathrm{e}^{2}-1\right)}{\mathrm{e}^{\frac{2}{n}}-1}\right]>\frac{\mathrm{e}}{2}\left(\mathrm{e}^{2}-1\right)$ <br> $\Rightarrow\frac{\mathrm{e}^{\frac{2}{n}}}{n\left(\mathrm{e}^{\frac{2}{n}}-1\right)}>\frac{1}{2}$ ( $\div\,\mathrm{e}\left(\mathrm{e}^{2}-1\right)$ on both sides since $\mathrm{e}\left(\mathrm{e}^{2}-1\right)>0$ ) <br> $\therefore k=\frac{1}{2}$ |