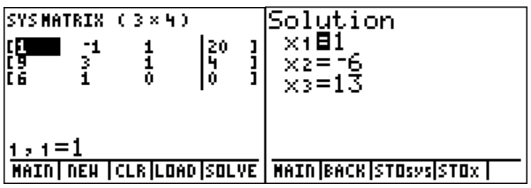
Suggested Solutions to 2014 SH2 H2 Mathematics Preliminary Examination Paper 1

1 (i)	Since $A(-1, 20)$ lies on the curve, $a(-1)^2 + b(-1) + c = 20 \Rightarrow a - b + c = 20$ – (1) Since $B(3, 4)$ lies on the curve, $a(3)^2 + b(3) + c = 4 \Rightarrow 9a + 3b + c = 4$ – (2) Since $B(3, 4)$ is a stationary point, $2a(3) + b = 0 \Rightarrow 6a + b = 0$ – (3)  $a = 1, b = -6, c = 13$
1 (ii)	$y = x^2 - 6x + 13 = (x-3)^2 + 4$ Hence $y = x^2$ $\downarrow$ translate 3 units in the positive x-direction $y = (x-3)^2$ $\downarrow$ translate 4 units in the positive y-direction $y = (x-3)^2 + 4 = x^2 - 6x + 13$
2	$\displaystyle\int \frac{x}{\left(1+4x^2\right)^2}\,\mathrm{d}x = \int \frac{1}{8}\cdot\frac{8x}{\left(1+4x^2\right)^2}\,\mathrm{d}x$ $\displaystyle= -\frac{1}{8}\left(\frac{1}{1+4x^2}\right) + c$ $\displaystyle\int \frac{4x^2}{\left(1+4x^2\right)^2}\,\mathrm{d}x = \int 4x\cdot\frac{x}{\left(1+4x^2\right)^2}\,\mathrm{d}x$ $\displaystyle= 4x\left(-\frac{1}{8\left(1+4x^2\right)}\right) - \int 4\left[-\frac{1}{8\left(1+4x^2\right)}\right]\mathrm{d}x$ $\displaystyle= -\frac{x}{2\left(1+4x^2\right)} + \int \frac{1}{2\left(1+4x^2\right)}\,\mathrm{d}x$ $\displaystyle= -\frac{x}{2\left(1+4x^2\right)} + \frac{1}{4}\tan^{-1} 2x + c$

3 (i)

$$x=\frac{t}{t^2-a}, \qquad y=te^{-t}$$

$$\frac{dx}{dt}=\frac{t^2-a-t(2t)}{\left(t^2-a\right)^2}=\frac{-t^2-a}{\left(t^2-a\right)^2}$$

$$\frac{dy}{dt}=-te^{-t}+e^{-t}=e^{-t}(1-t)$$

$$\frac{dy}{dx}=\frac{e^{-t}(1-t)\left(t^2-a\right)^2}{-t^2-a}$$

At $t=0$, $\dfrac{dy}{dx}=\dfrac{(-a)^2}{-a}=-a$.

Since the tangent to the curve C at $t = 0$ is perpendicular to the line $4y - x = 0$,

$\dfrac{1}{4}(-a)=-1 \Rightarrow a=4$ (Shown)

3 (ii)

For $a = 4$, $\dfrac{dy}{dx}=\dfrac{e^{-t}(1-t)\left(t^2-4\right)^2}{-t^2-4}$.

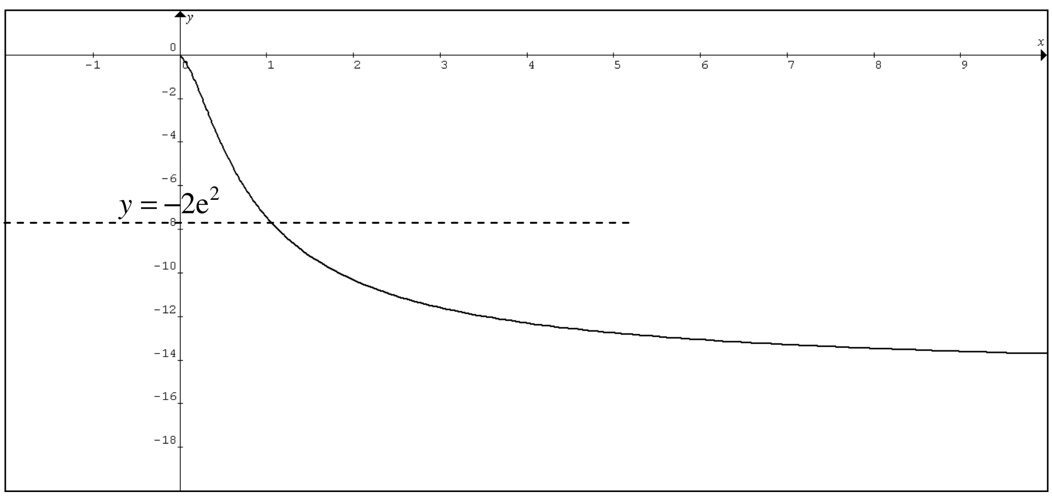
As $t\to -2$, $\dfrac{dy}{dx}\to\dfrac{e^2(3)(4-4)^2}{-4-4}=0$.

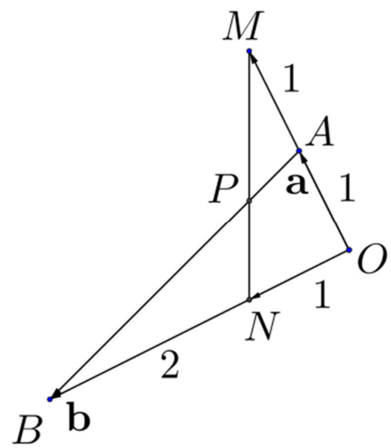
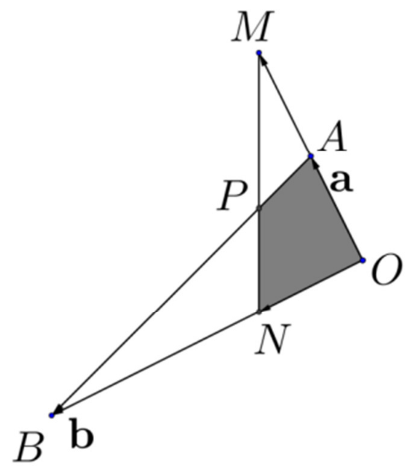
The gradient of the curve approaches 0 as $t\to -2$.

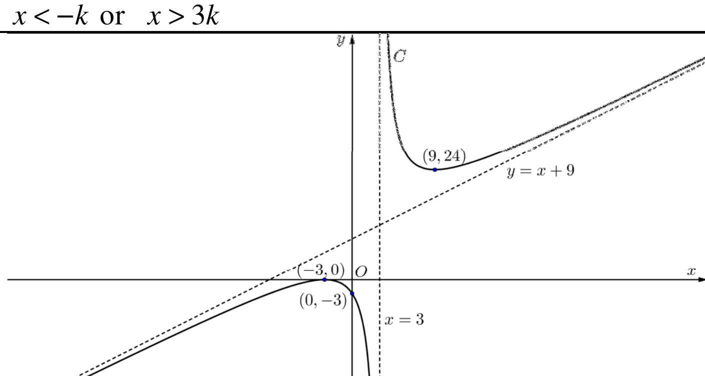
3 (iii)

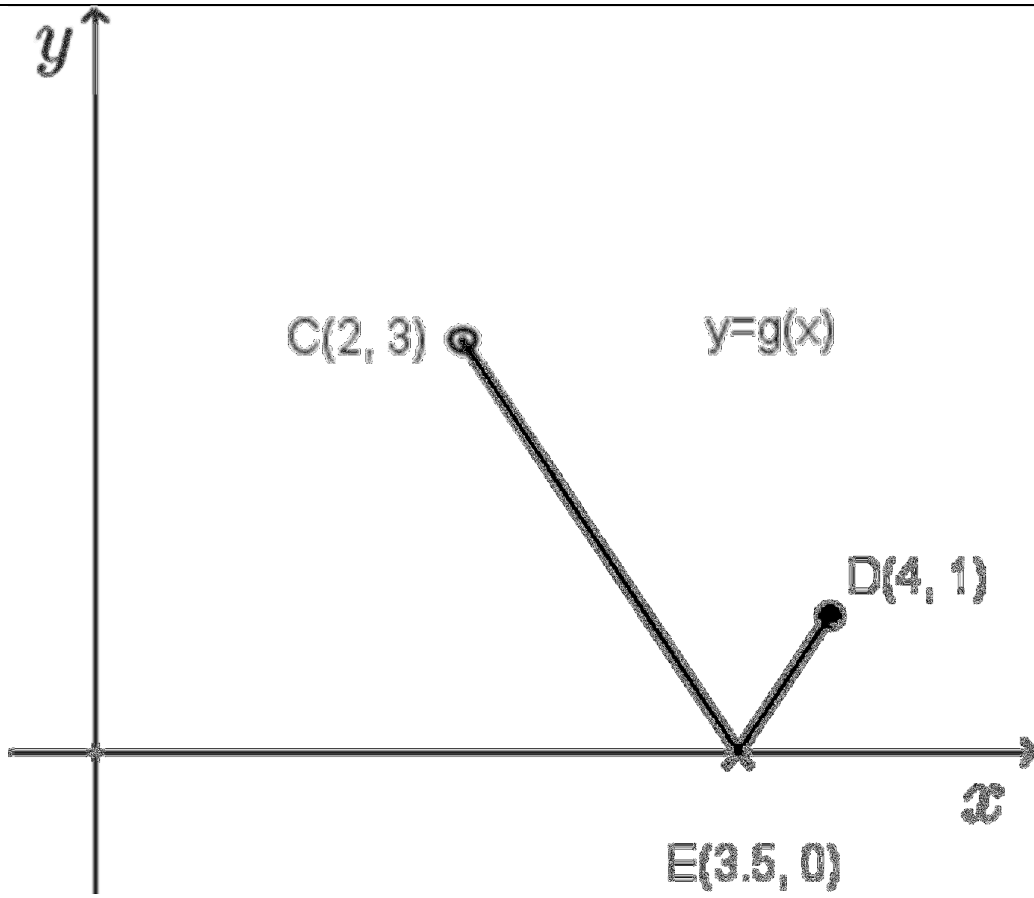
Observe that as $t\to -2$, $x=\dfrac{t}{t^2-4}\to\infty$, $\quad y=te^{-t}\to -2e^2$.

Thus, $y=-2e^2$ is a horizontal asymptote.

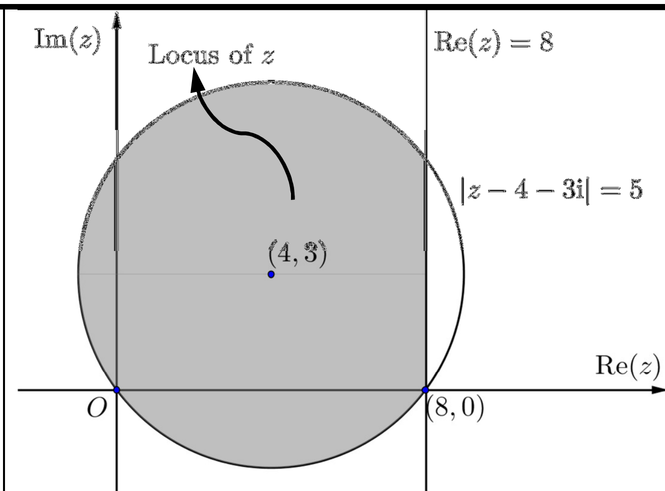


4 (i)	Since A, B and P are collinear, $\overrightarrow{OP} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b}$. Since M, N and P are collinear, $\overrightarrow{OP} = \mu 2\mathbf{a} + (1-\mu)\frac{1}{3}\mathbf{b}$. Comparing, we have $\begin{cases} \lambda = 2\mu \\ 1-\lambda = \frac{1}{3}(1-\mu) \end{cases}$. Solving, we have $\mu = \frac{2}{5}$ and $\lambda = \frac{4}{5}$. Thus, $\overrightarrow{OP} = \frac{4}{5}\mathbf{a} + \frac{1}{5}\mathbf{b} = \frac{1}{5}(4\mathbf{a}+\mathbf{b})$. 
4 (ii)	<u>Method 1</u>  Area of $OMN = \frac{1}{2}\left\|2\mathbf{a}\times\frac{1}{3}\mathbf{b}\right\| = \frac{1}{3}\|\mathbf{a}\times\mathbf{b}\|$ Area of $APM = \frac{1}{2}\left\|\mathbf{a}\times\left(\frac{1}{5}\mathbf{b}-\frac{1}{5}\mathbf{a}\right)\right\| = \frac{1}{10}\|\mathbf{a}\times\mathbf{b}\|$ Area of the quadrilateral $OAPN$ $= \frac{1}{3}\|\mathbf{a}\times\mathbf{b}\| - \frac{1}{10}\|\mathbf{a}\times\mathbf{b}\| = \frac{7}{30}\|\mathbf{a}\times\mathbf{b}\|$ <u>Method 2</u> Area of $OAB = \frac{1}{2}\|\mathbf{a}\times\mathbf{b}\|$ Area of $BPN = \frac{1}{2}\left\|\frac{2}{3}\mathbf{b}\times\left(\frac{4}{5}\mathbf{a}-\frac{4}{5}\mathbf{b}\right)\right\| = \frac{4}{15}\|\mathbf{a}\times\mathbf{b}\|$ Area of the quadrilateral $OAPN$ $= \frac{1}{2}\|\mathbf{a}\times\mathbf{b}\| - \frac{4}{15}\|\mathbf{a}\times\mathbf{b}\| = \frac{7}{30}\|\mathbf{a}\times\mathbf{b}\|$ <u>Method 3</u> Area of $OAP = \frac{1}{2}\left\|\mathbf{a}\times\frac{1}{5}(4\mathbf{a}+\mathbf{b})\right\| = \frac{1}{10}\|\mathbf{a}\times\mathbf{b}\|$ Area of ONP $= \frac{1}{2}\left\|\frac{1}{3}\mathbf{b}\times\frac{1}{5}(4\mathbf{a}+\mathbf{b})\right\|$ $= \frac{2}{15}\|\mathbf{a}\times\mathbf{b}\|$ Area of the quadrilateral $OAPN$ $= \frac{1}{10}\|\mathbf{a}\times\mathbf{b}\| + \frac{2}{15}\|\mathbf{a}\times\mathbf{b}\| = \frac{7}{30}\|\mathbf{a}\times\mathbf{b}\|$

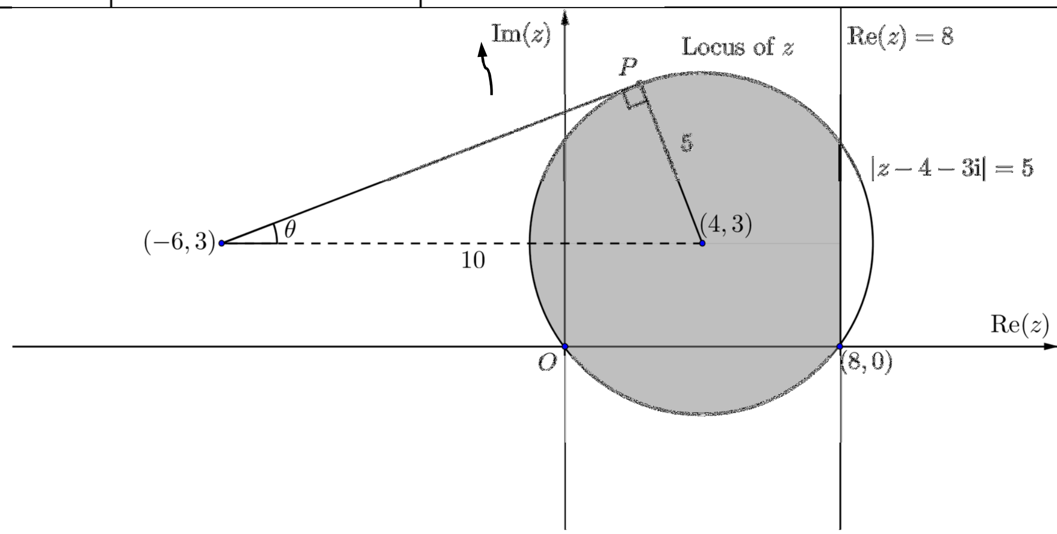
5 (i)	<u>Method 1</u> $\mathrm{f}(x)=\dfrac{(x+k)^2}{x-k},\ x\neq k$ $\Rightarrow \mathrm{f}'(x)=\dfrac{2(x+k)(x-k)-(x+k)^2}{(x-k)^2},\ x\neq k$ $=\dfrac{(x+k)(2x-2k-x-k)}{(x-3)^2},\ x\neq k$ $=\dfrac{(x+k)(x-3k)}{(x-k)^2},\ x\neq k$ For f to be increasing, $\mathrm{f}'(x)>0$ $\dfrac{(x+k)(x-3k)}{(x-k)^2}>0$ $(x+k)(x-3k)>0$ $x<-k$ or $x>3k$ <u>Method 2</u> $\mathrm{f}(x)=\dfrac{(x+k)^2}{x-k},\ x\neq k$ $=\dfrac{x^2+2kx+k^2}{x-k}$ $=\dfrac{x^2-kx+3kx-3k^2+4k^2}{x-k}$ $=x+3k+\dfrac{4k^2}{x-k}$ $\Rightarrow \mathrm{f}'(x)=1-\dfrac{4k^2}{\left(x-k\right)^2}$ For f to be increasing, $\mathrm{f}'(x)>0$ $1-\dfrac{4k^2}{\left(x-k\right)^2}>0$ $\dfrac{4k^2}{\left(x-k\right)^2}<1$ $4k^2<\left(x-k\right)^2$ $x-k<-2k$ or $x-k>2k$ $x<-k$ or $x>3k$
5 (ii)	

6 (i)	$y = \cos\left(\frac{\pi}{4}x\right) + 3.5$ $x = \left(\frac{4}{\pi}\right)\cos^{-1}(y - 3.5)$ $f^{-1}(x) = \left(\frac{4}{\pi}\right)\cos^{-1}(x - 3.5)$, for $x \in \mathbb{R}$, $\frac{5}{2} < x \le \frac{7-\sqrt{2}}{2}$ Range of f^{-1} is or [3, 4) or $3 \le x < 4$
6 (ii)	As range of f^{-1} is $[3, 4) \subseteq$ domain of g is (2, 4], gf^{-1} exists.
6 (iii)	 g(3) = 1, g(4) = 1, g(3.5) = 0 $D_{f^{-1}} \to R_{f^{-1}} = [3, 4) \to R_{gf^{-1}} = [0, 1]$ Range of gf^{-1} is [0, 1]

7 (i)



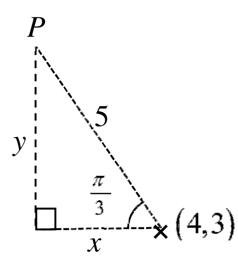
7 (ii)



$$\sin\theta = \frac{5}{10}$$

$$\theta = \frac{\pi}{6}$$

Thus, $\arg(z+6-3\mathrm{i}) = \frac{\pi}{6}$.



From the right angled triangle,

$$y = 5\sin\frac{\pi}{3} = \frac{5\sqrt{3}}{2}$$

$$x = 5\cos\frac{\pi}{3} = \frac{5}{2}$$

Thus the complex number z representing P is $\left(4-\frac{5}{2}\right)+\left(3+\frac{5\sqrt{3}}{2}\right)\mathrm{i}$, i.e.,

$\frac{3}{2}+\left(3+\frac{5\sqrt{3}}{2}\right)\mathrm{i}$.

8 (a) (i)	Total amount of prize fund needed $=1000+\left(\frac{4}{5}\right)(1000)+\left(\frac{4}{5}\right)^{2}(1000)+\cdots+\left(\frac{4}{5}\right)^{19}(1000)$ $=\dfrac{1000\left(1-\left(\frac{4}{5}\right)^{20}\right)}{1-\frac{4}{5}}$ $=5000\left(1-\left(\frac{4}{5}\right)^{20}\right)=4942.35\ (\text{nearest cents})$
8 (a)(ii)	Assume that no two athletes will arrive at the same time.
8 (a) part after (ii)	$\frac{n}{2}\left[2(15)+(n-1)(185)\right]\leq 100000$ $n\left[30+185n-185\right]\leq 200000$ $185n^{2}-155n-200000\leq 0$ By GC, $-32.4\leq n\leq 33.3$. Maximum n is 33.
8 (b)	Given $S_n=10-\dfrac{2^{n+1}}{5^{n-1}}$, $T_n=S_n-S_{n-1}=10-\dfrac{2^{n+1}}{5^{n-1}}-\left(10-\dfrac{2^{n}}{5^{n-2}}\right)$ $=\dfrac{2^{n}}{5^{n-2}}-\dfrac{2^{n+1}}{5^{n-1}}$ $=\dfrac{25(2^{n})}{5^{n}}-\dfrac{5(2^{n}.2)}{5^{n}}$ $=\dfrac{15(2^{n})}{5^{n}}$, for $n\geq 2$ $T_1=S_1=10-\dfrac{2^{1+1}}{5^{1-1}}$ $=6=\dfrac{15(2^{1})}{5^{1}}$ $\therefore T_n=\dfrac{15(2^{n})}{5^{n}}$, for all $n\in\mathbb{Z}^{+}$ $\dfrac{T_n}{T_{n-1}}=\dfrac{15(2^{n})}{5^{n}}\div\dfrac{15(2^{n-1})}{5^{n-1}}$ $=\dfrac{15(2^{n})}{5^{n}}\times\dfrac{5^{n-1}}{15(2^{n-1})}=\dfrac{2^{n}}{5^{n}}\times\dfrac{2.5^{n}}{5.2^{n}}=\dfrac{2}{5}$ Since $\dfrac{T_n}{T_{n-1}}$ gives a constant, the series is a geometric progression. Common ratio is 2/5.

9 (a)	$AD^2 = 5^2+6^2-2(5)(6)\cos(\alpha-\theta)$ $= 61-60(\cos\theta\cos\alpha+\sin\theta\sin\alpha)$ $= 61-60\left(\frac{3}{5}\cos\theta+\frac{4}{5}\sin\theta\right)$ $= 61-36\cos\theta-48\sin\theta$ $\approx 61-36\left(1-\frac{\theta^2}{2}\right)-48\theta$ $= 25-48\theta+18\theta^2$ $AD \approx (25-48\theta+18\theta^2)^{\frac{1}{2}}$ $= 5\left(1-\frac{48}{25}\theta+\frac{18}{25}\theta^2\right)^{\frac{1}{2}}$ $\approx 5\left[1+\frac{1}{2}\left(-\frac{48}{25}\theta+\frac{18}{25}\theta^2\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(-\frac{48}{25}\theta\right)^2\right]$ $= 5\left(1-\frac{24}{25}\theta-\frac{63}{625}\theta^2\right)$ $= 5-\frac{24}{5}\theta-\frac{63}{125}\theta^2$
9 (b)	$\mathrm{f}(x) = \mathrm{e}^{\sin^{-1}nx} \Rightarrow \mathrm{f}(0) = 1$ $\mathrm{f}'(x) = \frac{n}{\sqrt{1-(nx)^2}}\mathrm{e}^{\sin^{-1}nx} = \frac{n}{\sqrt{1-n^2x^2}}\mathrm{f}(x) \Rightarrow \mathrm{f}'(0) = n$ $\mathrm{f}''(x) = \frac{n^3x}{\left(1-(nx)^2\right)^{3/2}}\mathrm{f}(x)+\frac{n}{\sqrt{1-n^2x^2}}\mathrm{f}'(x) \Rightarrow \mathrm{f}''(0) = n^2$ $\mathrm{f}(x) = 1+nx+\frac{n^2}{2}x^2$ Hence n = 2 and $b = \frac{2^2}{2} = 2$.

10 (i)	Let P_n be statement "$u_n = \dfrac{3n}{n+2}$, for $n \in \mathbb{Z}^+$". LHS of $P_1 = u_1 = 1$ and RHS of $P_1 = \dfrac{3(1)}{1+2} = 1 =$ LHS of P_1 Hence, P_1 is true. Assume that P_k is true, i.e. $u_k = \dfrac{3k}{k+2}$ for some $k \in \mathbb{Z}^+$. Consider P_{k+1} i.e. $u_{k+1} = \dfrac{3(k+1)}{(k+1)+2}$. LHS of P_{k+1}= $u_{k+1} = u_k + \dfrac{6}{(k+2)(k+3)}$ $= \dfrac{3k}{k+2} + \dfrac{6}{(k+2)(k+3)}$ $= \dfrac{3k(k+3)}{(k+2)(k+3)} + \dfrac{6}{(k+2)(k+3)}$ $= \dfrac{3k^2+9k+6}{(k+2)(k+3)} = \dfrac{3(k^2+3k+2)}{(k+2)(k+3)}$ $= \dfrac{3(k+1)(k+2)}{(k+2)(k+3)} = \dfrac{3(k+1)}{(k+1)+2}$ = (RHS) Thus, P_k is true $\Rightarrow$ P_{k+1} is true. Since P_1 is true and P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$.
10 (ii)	$= \sum_{n=1}^{N} \dfrac{6}{(n+2)(n+3)}$ $= u_2 - u_1$ $+ u_3 - u_2$ $+ \ldots$ $+ u_N - u_{N-1}$ $+ u_{N+1} - u_N$ $= u_{N+1} - u_1$ $= \dfrac{3(N+1)}{N+3} - 1$ $= \dfrac{2N}{N+3}$ or $2 - \dfrac{6}{N+3}$
10 (iii)	$\sum_{n=10}^{\infty} \dfrac{3}{(n+2)(n+3)}$ $= \dfrac{1}{2}\left[\sum_{n=1}^{\infty} \dfrac{6}{(n+2)(n+3)} - \sum_{n=1}^{9} \dfrac{6}{(n+2)(n+3)}\right]$ $= \lim_{N\to\infty}\left(1 - \dfrac{3}{N+3}\right) - \dfrac{9}{9+3} = 1 - \dfrac{9}{12} = \dfrac{1}{4}$

10 (iv)	Let $j - 3 = n + 2$ $\Rightarrow j = n + 5$ $\sum_{j=6}^{N} \frac{6}{(j-2)(j-3)}$ $= \sum_{j=6}^{j=N} \frac{6}{(j-2)(j-3)}$ $= \sum_{n+5=6}^{n+5=N} \frac{6}{(n+5-2)(n+5-3)}$ $= \sum_{n=1}^{N-5} \frac{6}{(n+2)(n+3)} = \frac{2N-10}{N-2}$

11 (a)	Method 1 $z = e^{2x} y$ $y = z e^{-2x}$ $\frac{dy}{dx} = \frac{dz}{dx} e^{-2x} - 2z e^{-2x}$ $\frac{dy}{dx} + 2y = (x+1) e^{x^2}$ $\frac{dz}{dx} e^{-2x} - 2z e^{-2x} + 2z e^{-2x} = (x+1) e^{x^2}$ $\frac{dz}{dx} = (x+1) e^{x^2+2x}$ Method 2 $z = e^{2x} y$ $\frac{dz}{dx} = e^{2x} \frac{dy}{dx} + 2e^{2x} y = e^{2x} \left(\frac{dy}{dx} + 2y \right)$ $\frac{dy}{dx} + 2y = (x+1) e^{x^2}$ $e^{2x} \left(\frac{dy}{dx} + 2y \right) = (x+1) e^{x^2+2x}$ $\frac{dz}{dx} = (x+1) e^{x^2+2x}$
	$\frac{dz}{dx} = (x+1) e^{x^2+2x}$ $\frac{dz}{dx} = \frac{1}{2}(2x+2) e^{x^2+2x}$ $z = \int \frac{1}{2}(2x+2) e^{x^2+2x} \, dx$ $= \frac{1}{2} e^{x^2+2x} + c$, where c is an arbitrary constant $y e^{2x} = \frac{1}{2} e^{x^2+2x} + c$ $y = \frac{1}{2} e^{x^2} + c e^{-2x}$

11 (b) (i)	$\frac{d\theta}{dt} = -k(\theta - 20)$ As $\frac{d\theta}{dt} = -1$ when $\theta = 70$, $-1 = -k(70-20) \Rightarrow k = 0.02$ $\frac{d\theta}{dt} = -0.02(\theta - 20)$ When $\theta = 40$, $\frac{d\theta}{dt} = -0.02(40-20) = -0.4$ It is cooling at a rate of 0.4°C per minute.
	$\frac{d\theta}{dt} = -0.02(\theta - 20)$ $\frac{1}{\theta - 20}\frac{d\theta}{dt} = -0.02$ $\int \frac{1}{\theta - 20}\frac{d\theta}{dt}\, dt = \int -0.02\, dt$ $\ln\lvert\theta - 20\rvert = -0.02t + c$ $\lvert\theta - 20\rvert = e^{-0.02t+c}$ $\theta - 20 = \pm e^{-0.02t+c}$ $= Ae^{-0.02t}$, where $A = \pm e^{c}$ $\theta = 20 + Ae^{-0.02t}$ Given $\theta = 95$ when $t = 0$, $95 = 20 + A(1)$ $A = 75$ $\theta = 20 + 75e^{-0.02t}$
11(b) (ii)	It is a good model because the equation reflects the decrease of the temperature to a steady state temperature, which is what would happen in real life.

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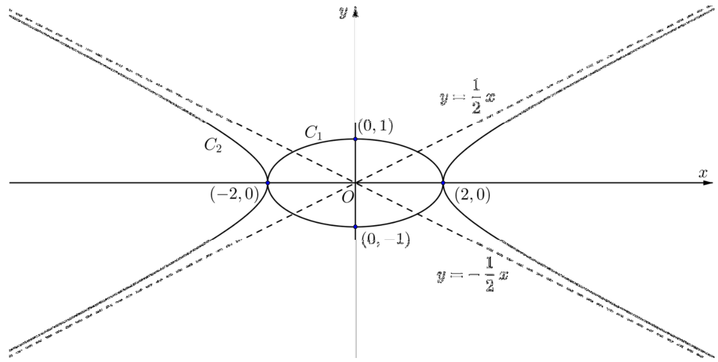

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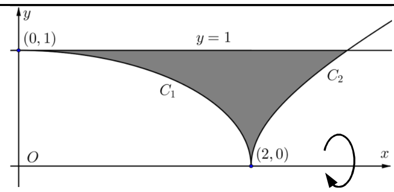

$C_1: \quad \frac{x^2}{4}+y^2=1 \Rightarrow y^2=1-\frac{x^2}{4}$

$C_2: \quad \frac{x^2}{4}-y^2=1 \Rightarrow y^2=\frac{x^2}{4}-1$

Volume generated

$=\pi\left(1^2\right)\left(\sqrt{8}\right)-\pi\int_0^2\left(1-\frac{x^2}{4}\right)\mathrm{d}x-\pi\int_2^{\sqrt{8}}\left(\frac{x^2}{4}-1\right)\mathrm{d}x$

$=3.4701$ (to 4 dec places)
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<tr><td>12 (iii)</td><td>
$\frac{x^2}{4}+y^2=1 \Rightarrow y=\sqrt{1-\frac{x^2}{4}}$

Using the substitution $x=2\cos\theta$, $\frac{\mathrm{d}x}{\mathrm{d}\theta}=-2\sin\theta$.

When $x=1, \theta=\frac{\pi}{3}$; when $x=2, \theta=0$.

Area of region $=\int_1^2\sqrt{1-\frac{x^2}{4}}\,\mathrm{d}x=\int_{\frac{\pi}{3}}^0\sqrt{1-\frac{4\cos^2\theta}{4}}\left(-2\sin\theta\right)\,\mathrm{d}\theta$

$=\int_0^{\frac{\pi}{3}} 2\sin^2\theta\,\mathrm{d}\theta=\int_0^{\frac{\pi}{3}} 1-\cos 2\theta\,\mathrm{d}\theta$

$=\left[\theta-\frac{1}{2}\sin\left(2\theta\right)\right]_0^{\frac{\pi}{3}}$

$=\left[\frac{\pi}{3}-\frac{1}{2}\sin\left(\frac{2\pi}{3}\right)\right]-\left[0-\frac{1}{2}\sin\left(0\right)\right]$

$=\frac{\pi}{3}-\frac{\sqrt{3}}{4}$
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