MERIDIAN JUNIOR COLLEGE
JC2 Preliminary Examination
Higher 2

H2 Mathematics 9740/01

Paper 1 **16 September 2014**

3 Hours

Additional Materials: Writing paper
List of Formulae (MF 15)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 Use the method of mathematical induction to prove that $\sum_{r=1}^{n}\frac{r-1}{3^r}=\frac{1}{4}-\frac{1}{4}\left(\frac{1}{3}\right)^n(2n+1)$. [5]

2 The function f is defined by

$$\mathrm{f}: x \mapsto \ln\left(\frac{1}{1+x}\right), \quad x \in \mathbb{R},\ -1 < x \leq 1.$$

(i) Show that f^{-1} exists. [1]

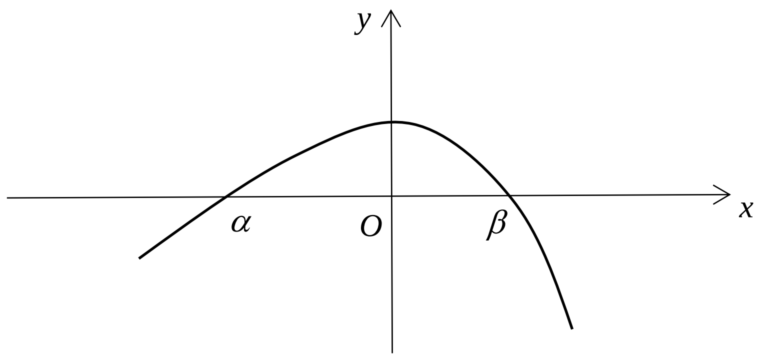
(ii) Define f^{-1} in a similar form. [3]

(iii) Find an expression for $\mathrm{g}(x)$ for each of the following cases:

(a) $\mathrm{fg}(x)=x$ [1]

(b) $\mathrm{gf}(x)=(x+1)^2$ [2]

3



The diagram shows the graph of $y=x+2-\mathrm{e}^x$. The two roots of the equation $x+2-\mathrm{e}^x=0$ are denoted by α and β where $\alpha<0$ and $\beta>0$.

(i) Find the values of α and β correct to 4 decimal places. [2]

A sequence of positive real numbers $x_1, x_2, x_3, ...$ satisfies the recurrence relation

$$x_{n+1}=\ln(x_n+2)$$

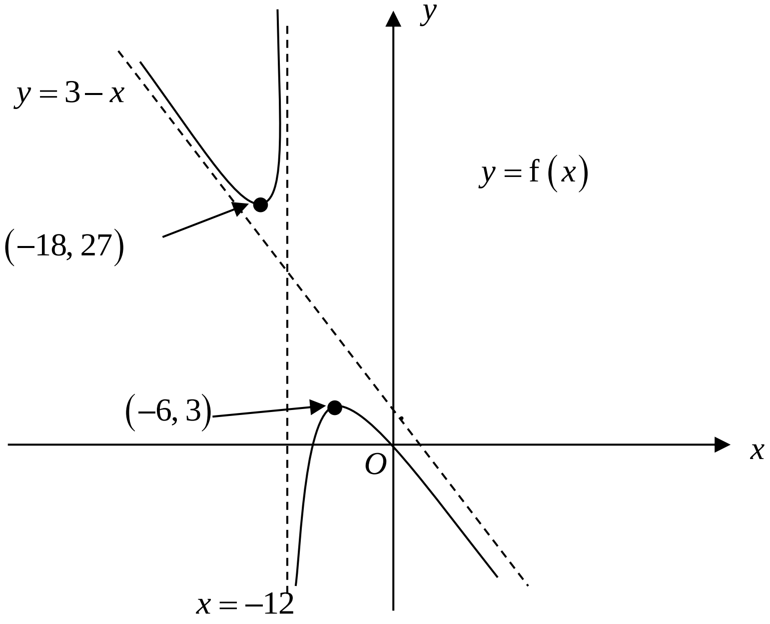
for $n \geq 1$.

(ii) Prove algebraically that, if the sequence converges, then it converges to β. [2]

(iii) Use a calculator to determine the behaviour of the sequence when $x_1=2$. [1]

(iv) By referring to the graph above, prove that $x_{n+1}<x_n$ if $x_n>\beta$. [2]

4 The diagram below shows the graph of $y = \mathrm{f}(x)$.



The curve passes through the origin and has turning points at $(-18, 27)$ and $(-6, 3)$. It is given further that the curve intersects the line $y = 2$ at $x = -8$ and $x = -3$. On separate clearly labelled diagrams, sketch the graphs of

(i) $y = \mathrm{f}(ax) - 2$, [3]

(ii) $y = \dfrac{1}{\mathrm{f}(ax) - 2}$, [3]

where *a* is a positive real constant.

When $m > k$, $\mathrm{f}(ax) - 2 = 1 - mx$ has two real and distinct roots. Hence or otherwise, state the least value of *k* in terms of *a*. [1]

5 Given that $y = \cos^{-1}(2x)$, prove that $\sin y \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 0$. [3]

(i) By further differentiation of this result, find the series expansion of *y* in ascending powers of *x* up to and including the term in x^3. Give the coefficients in exact form. [3]

(ii) Hence find the series expansion of $\dfrac{\cos^{-1}(2x)}{\sqrt{1-3x^2}}$ in ascending powers of *x* up to and including the term in x^2. Give the coefficients in exact form. [2]

[Turn Over

6 The number of trees in a forest after t years is x. The number of trees increases at a rate of $\dfrac{x-k}{t}$ per year where k is the initial number of trees. Trees are chopped off at a rate of $\dfrac{5000}{t+5}$ per year. In addition, environmentalists also plant 1000 new trees every year.

(i) Write down a differential equation to model the population of trees in the forest. [1]

(ii) By using the substitution $y = \dfrac{x-k}{t}$, show that the above differential equation can be reduced to

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1000}{t+5}.$$ [2]

(iii) Given that the number of trees returns to k after 10 years, express x in terms of t and k. Hence by considering the graph of x against t or otherwise, deduce the minimum initial number of trees (to the nearest integer) such that the trees will not become extinct. [6]

7 The equations of three planes p_1, p_2, and p_3 are

$$x - 5y + 2z = 13,$$
$$2x - y - 5z = -1,$$
$$\lambda x - 4y + z = \mu,$$

respectively, where λ and μ are constants.

(i) Find the acute angle between the planes p_1 and p_2. [2]

The planes p_1 and p_2 intersect in a line l.

(ii) Find a vector equation of l. [2]

(iii) Given that all three planes meet in the line l, find λ and μ. [3]

(iv) Given instead that the plane p_3 intersects the planes p_1 and p_2 at distinct lines m_1 and m_2 respectively, such that m_1 is parallel to m_2, describe the geometrical relationship of the planes p_1, p_2 and p_3. Hence comment on the values of λ and μ. [3]

8 **(a)** Solve the equation $z^5+2-2\mathrm{i}=0$, giving the roots in the form $r\mathrm{e}^{\mathrm{i}\alpha}$, where $r>0$ and $-\pi<\alpha\le\pi$. [4]

Show the roots on an Argand diagram. [2]

(b) Show that $\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}-\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}}=2\mathrm{i}\sin\frac{\theta}{2}$. Hence show that $\dfrac{\mathrm{e}^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\theta}}=\dfrac{1}{2}\left(\mathrm{i}\cot\dfrac{\theta}{2}-1\right)$. [4]

9 It is given that $\mathrm{f}(x)=2-\dfrac{1}{x-1}$.

(i) On separate diagrams, sketch the graphs of $y=\mathrm{f}(|x|)$ and $y=|\mathrm{f}(x)|$, giving the coordinates of any points where the graphs meet the *x*- and *y*- axes. You should label the graphs clearly. [4]

(ii) Use a non-calculator method to find the area bounded by $y=\mathrm{f}(|x|)$, $x=\dfrac{5}{4}$, $x=2$ and the *x*-axis. [4]

(iii) The region R is bounded by $y=|\mathrm{f}(x)|$, $y=2$ and $x=\dfrac{3}{2}$. Find the volume of revolution formed when R is rotated completely about the *y*-axis. [4]

10 The parametric equations of a curve C are

$$x=\cos^3\theta,\qquad y=2\sin^3\theta\quad\text{where } -\frac{\pi}{2}\le\theta\le 0.$$

(i) Sketch the graph of C. [2]

(ii) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of θ. [2]

(iii) The tangent to the curve C at the point $P\left(\cos^3 t,\ 2\sin^3 t\right)$ intersects the *x*-axis and the *y*-axis at the points U and V respectively. Show that the coordinates of U are $(a\cos t,\ 0)$, where a is a constant to be found. Find the coordinates of V. [5]

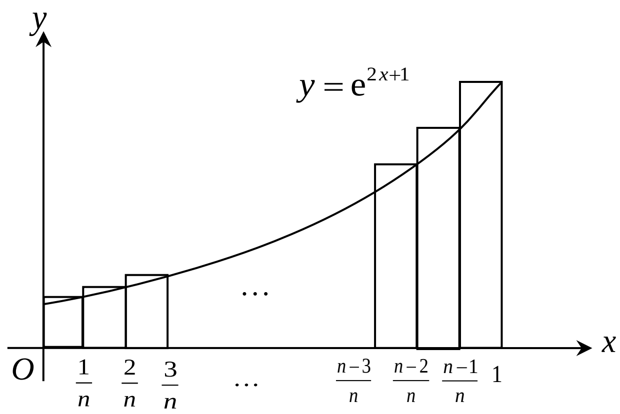
(iv) Find a cartesian equation of the locus of the mid-point of UV as t varies. [3]

11 **(a)** **(i)** Show that $(k+1)!k - k!(k-1) = k!(k^2+1)$. [1]

(ii) Hence find $\sum_{k=1}^{n} k!(k^2+1)$. [3]

(iii) Using your answer in part **(ii)**, find $\sum_{k=1}^{n-1} (k+1)!(k^2+2k+2)$. [3]

(b) The graph of $y = e^{2x+1}$, for $0 \le x \le 1$, is shown in the diagram. Rectangles of equal width are drawn as shown in the interval between $x = 0$ and $x = 1$.



(i) Show that the total area of all the n rectangles, A, is given by $\frac{e}{n}\left(\frac{e^{\frac{2}{n}}\left(1-e^2\right)}{1-e^{\frac{2}{n}}}\right)$. [2]

(ii) By considering the area under the curve $y = e^{2x+1}$, find the exact value of the limit of A as $n \to \infty$. [2]

(iii) Hence show that $\frac{e^{\frac{2}{n}}}{n\left(e^{\frac{2}{n}}-1\right)} > k$, where k is a constant to be found. Find the largest possible value of k. [2]

END OF PAPER