Class Adm No

Candidate Name: ______________________

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millennia institute

2014 Preliminary Examination II
Pre-University 3

MATHEMATICS **9740/01**

Paper 1 **17 September 2014**

3 hours

Additional Materials: Answer Paper
List of Formulae (MF 15)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, arrange your answers in NUMERICAL ORDER and fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

Answer all the questions [100 marks]

1. The lines l_1 and l_2 have equations

$$\mathbf{r}=\begin{pmatrix}0\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}-2\\1\\-1\end{pmatrix} \text{ and } \mathbf{r}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

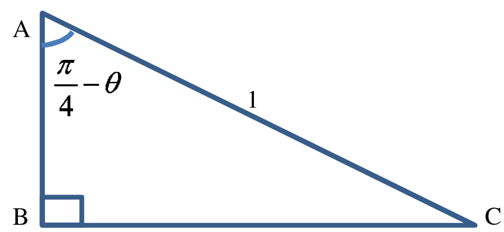
respectively.

(i) Show that l_1 and l_2 are skew lines. **[2]**

(ii) Find the acute angle between the lines l_1 and l_2. **[2]**

2. **(i)** Show that $\sin(A+B)+\sin(A-B)=2\sin A\cos B$. **[1]**

(ii)



In the triangle ABC, AC = 1, angle ABC = $\frac{\pi}{2}$ radians and angle BAC = $\left(\frac{\pi}{4}-\theta\right)$ radians (see diagram). Given that θ is sufficiently small for θ^5 and higher powers to be neglected, by using the result in part **(i)**, show that

$$AB+BC\approx p\left(1-\frac{\theta^2}{2}+\frac{\theta^4}{q}\right),$$

where p and q are constants to be determined. **[4]**

3. As soon as an organism dies, the rate at which Carbon-14 will decay is given as $\frac{\mathrm{d}x}{\mathrm{d}t}=-kx$, where k is a positive constant, x is the amount of Carbon-14 present in the organism and t is the time in years.

(i) Assuming that there is x_0 grams of Carbon-14 initially, show that the time taken for the amount of Carbon-14 in the organism to be halved is $\frac{\ln 2}{k}$ years. **[3]**

(ii) The organism in part **(i)** is found to contain 0.2% of its original Carbon-14. Given that the time taken for the amount of Carbon-14 in the organism to be halved is 5730 years, how long has the organism been left to decay? **[3]**

4. Referred to the origin O, the position vectors of the points A and C are $\mathbf{a}$ and $\mathbf{c}$ respectively, and $OABC$ is a parallelogram.

Express $\overrightarrow{OB}$ and $\overrightarrow{AC}$ in terms of $\mathbf{a}$ and $\mathbf{c}$. **[1]**

Prove that $\left|\overrightarrow{OB}\right|^2 + \left|\overrightarrow{AC}\right|^2 = \left|\overrightarrow{OA}\right|^2 + \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{CO}\right|^2$. **[3]**

Using the result above, what can be said about $\mathbf{a}$ and $\mathbf{c}$ when $\left|\overrightarrow{OB}\right| = \left|\overrightarrow{AC}\right|$? **[2]**

5. **(a)** **(i)** Given that T_n represents the n^{th} term of an arithmetic progression, show that the terms of a new sequence where the n^{th} term is represented by $U_n = e^{T_n}$ forms a geometric progression. **[2]**

(ii) Given that X_n represents the n^{th} term of a geometric progression, show that the terms of a new sequence where the n^{th} term is represented by $Y_n = \ln X_n$ forms an arithmetic progression. **[2]**

(b) The first, third and fourth terms of an arithmetic progression are consecutive terms of a geometric progression. Show that $a = -4d$, where a and d represent the first term and common difference of the arithmetic progression respectively. **[2]**

Hence determine if the geometric progression is convergent. **[1]**

6. **(i)** Find the general solution $y = \mathrm{f}(x)$ of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = ae^{2x} + be^{-3x}.$$ **[2]**

(ii) Given that the Maclaurin series for $y = \mathrm{f}(x)$, up to and including the term in x^2, is $y = 2 - x + \frac{13}{2}x^2$, find the particular solution of the differential equation in **(i)** for which the solution curve passes through the point $\left(\ln 2, \frac{33}{8}\right)$. **[5]**

7. **(a)** Find $\int \tan^{-1} x \, \mathrm{d}x$. **[3]**

(b) Find $\int \frac{2x}{x^2 + 2x + 1} \, \mathrm{d}x$. **[3]**

(c) Use the substitution $x = \frac{1}{u}$ to find the exact value of $\int_1^2 \frac{1}{x\sqrt{x^2 - 1}} \, \mathrm{d}x$. **[4]**

[Turn over

8. A curve C is defined by the parametric equations $x = \cos t$, $y = \sin 2t$, for $0 \le t \le 2\pi$.

(i) Sketch the curve, stating the coordinates of any points of intersection with the axes. **[2]**

(ii) Show that the area enclosed by the curve C is $8\int_0^{\frac{\pi}{2}} \sin^2 t \cos t \, \mathrm{d}t$.

Given that $\int \sin^2 t \cos t \, \mathrm{d}t = \frac{1}{3}\sin^3 t + c$, find the exact area enclosed by the curve C. **[4]**

(iii) By finding the cartesian equation of C or otherwise, find the volume of revolution formed when the area enclosed by the curve C is rotated completely about the x-axis, giving your answer correct to 3 decimal places. **[3]**

9. The curves C_1 and C_2 have equations $y = \frac{x-1}{x-2}$ and $y^2 = \frac{x-1}{x-2}$ respectively.

(i) Sketch C_1 and C_2 on **separate** diagrams, stating the coordinates of any points of intersection with the axes and the equations of any asymptotes. **[6]**

(ii) Show algebraically that the x-coordinates of the points of intersection of C_1 and $y = x^2$ satisfy the equation $x^3 - 2x^2 - x + 1 = 0$.

By including an addition graph in the sketch of C_1, solve $x^3 - 2x^2 - x + 1 = 0$. **[3]**

(iii) Deduce the x-coordinates of the points of intersection of C_2 and $y = x$. **[1]**

10. **(i)** By using de Moivre's theorem, or otherwise, show that

$$(1-i)^n = 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right). \quad \textbf{[2]}$$

(ii) Using the result in **(i)** or otherwise, find the least positive integer n for which $(1-i)^n$ is real and negative. Solution by trial and error will not be accepted. **[3]**

(iii) For the equation $z^4 + az^3 + bz^2 + cz + d = 0$ where a, b, c and d are real, give a brief explanation and determine the possible number of complex roots the equation can have. **[2]**

(iv) Solve the equation $z^4 + 4 = 0$, expressing the solutions in the form $x + iy$ where x and y are real. **[4]**

11. A sequence of real numbers $x_1, x_2, x_3, \ldots$ satisfies the recurrence relation

$$x_1 = 1, \quad x_2 = 1, \quad x_{r+1} = 2x_r + x_{r-1} \quad \text{for } r \geq 2.$$

(i) Show that $\dfrac{1}{x_{r-1}x_r} - \dfrac{1}{x_r x_{r+1}} = \dfrac{2}{x_{r-1}x_{r+1}}$. **[2]**

(ii) Hence show that $\displaystyle\sum_{r=2}^{n} \frac{1}{x_{r-1}x_{r+1}} = \frac{1}{2}\left(1 - \frac{1}{x_n x_{n+1}}\right)$. **[3]**

(iii) Prove the result in part **(ii)** by mathematical induction. **[5]**

(iv) Find the sum to infinity of the series in part **(ii)**. **[2]**

12. Referred to an origin O, the position vector of A is $2\mathbf{i} - \mathbf{k}$ and the equation of line l is $\mathbf{r} = -7\mathbf{i} + 15\mathbf{j} - 5\mathbf{k} + \lambda(3\mathbf{i} - 7\mathbf{j} + 4\mathbf{k})$.

(i) Find the position vectors of B and C, both lying on l, such that $AB = AC = 10$. **[3]**

(ii) Given that M is the midpoint of BC and the plane π_1 contains l and is perpendicular to AM, show that the equation of the plane π_1 is $-3x + y + 4z = 16$. **[4]**

(iii) The planes π_2 and π_3 have equations $x + 2y - 3z = 5$ and $x - 2y + z = 1$ respectively. Verify that A lies in both π_2 and π_3. **[1]**

(iv) Determine the position vector of D, the point of intersection between π_1, π_2 and π_3. **[2]**

(v) Hence, or otherwise, find the volume of the tetrahedron $ABCD$. **[3]**

[Volume of tetrahedron = $\dfrac{1}{3} \times$ area of triangular base $\times$ perpendicular height]

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