JURONG JUNIOR COLLEGE

Preliminary Examinations

MATHEMATICS
Higher 2

9740/01

2 September 2014

Paper 1

3 hours

Additional materials: Answer Paper
Cover Page
List of Formulae (MF 15)

READ THESE INSTRUCTIONS FIRST

Write your name and civics class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together, with the cover page in front.

This document consists of **6** printed pages.

1 In a “catch the vouchers” game, the player has to enter into a small confined room where there are many different colour vouchers flying in the air. The player has only 30 seconds to catch as many vouchers as possible. Different colour vouchers carry different value of money and same colour vouchers carry the same value of money. Four players play the game. The number of different colour vouchers caught and the total amount of money won by each player are shown in the table below.

	Player A	Player B	Player C	Player D
Blue	3	5	4	2
Yellow	4	2	p	8
Red	7	4	2	5
Total amount ($)	$27.40	$20.80	$43.40	$45.00

Find p. [4]

2 A sequence u_0, u_1, u_2, …is defined by $u_0 = 3$ and $u_{n+1} = 1 - 2u_n$, where $n \geq 0$.

(i) Prove by induction that $u_n = \frac{1}{3}\left[1 + 8\left(-2\right)^n\right]$, for all $n \geq 0$. [4]

(ii) State, briefly giving a reason for your answer, whether the sequence is convergent. [1]

3 Find integers a and b such that

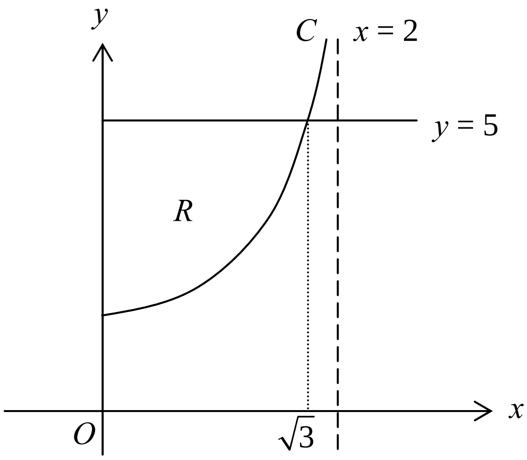
$$(r + 1)^4 + (r + 1)^2 + 1 \equiv (r^2 + ar + 3)(r^2 + r + b).$$ [2]

With these values of a and b, and by considering $\frac{1}{r^2 + r + b} - \frac{1}{r^2 + ar + 3}$,

find $\sum_{r=0}^{N} \frac{r+1}{(r+1)^4 + (r+1)^2 + 1}$ in terms of N. [4]

Deduce $\sum_{r=2}^{N} \frac{r}{r^4 + r^2 + 1}$ in terms of N. [2]

4 The diagram shows a region R in the first quadrant bounded by the curve C with equation $y = \dfrac{2}{\sqrt{4-x^2}} + 3$, the y-axis and the line $y = 5$. The line $y = 5$ and the curve intersect at the point $\left(\sqrt{3}, 5\right)$.

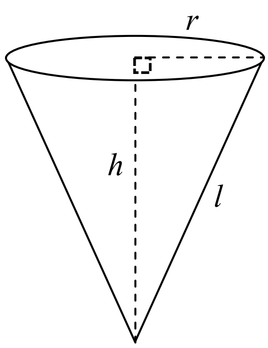


(i) Calculate the area of region R. [2]

(ii) Write down the equation of the curve obtained when C is translated by 5 units in the negative y-direction. [1]

Hence show that the volume of the solid formed when R is rotated completely about the line $y = 5$ is given by $4\pi \displaystyle\int_0^{\sqrt{3}} \left(\frac{1}{4-x^2} - \frac{2}{\sqrt{4-x^2}} + 1 \right) dx$, and evaluate this integral exactly. [4]

5 [It is given that a cone of radius r, height h and slant length l has volume $\frac{1}{3}\pi r^2 h$ and curved surface area $\pi r l$.]



An ice cream cone wafer (as shown in the diagram above) of negligible thickness is to have a fixed external surface area of $k\pi \text{ cm}^2$. Show that the volume V of the cone is given by

$$V = \frac{\pi r \sqrt{k^2 - r^4}}{3}.$$

Use differentiation to find the radius r cm of the cone in terms of k that will give a minimum internal volume of the cone (you need not prove the minimum value of V). [6]

6 **(a)** **(i)** Obtain a formula for $\int_1^n \frac{1}{x^2} \ln x \, dx$ in terms of n, where $n > 1$. [3]

(ii) Hence evaluate $\int_1^\infty \frac{1}{x^2} \ln x \, dx$. [1]

[You may assume that $\frac{1}{n} \ln n \to 0$ as $n \to \infty$.]

(b) Use the substitution $x = a\sec\theta$ to find the exact value of $\int_a^{2a} \frac{\sqrt{x^2 - a^2}}{x} dx$ in terms of a and π, where a is a positive constant . [4]

7 An athlete hopes to represent Singapore at the SEA Games in 2015 and he embarks on a rigorous training programme.

For his first training session, he ran a distance of 7.5 km. For his subsequent training sessions, he ran a distance of 800 m more than the previous training session.

(i) Express, in terms of n, the distance (in km) he ran on his n th training session. [1]

(ii) Find the minimum number of training sessions required for him to run a total distance of at least 475 km. [3]

After a month, he realised that his progress was unsatisfactory and he decided to modify the training. For the modified training programme, he ran a distance of x km for the first session, and on each subsequent training session, the distance covered is $\frac{6}{5}$ times of the previous session.

(iii) Find x, to the nearest integer, if he covered a distance of 14.93 km on the 6th training session. [2]

(iv) Using the answer in **(iii)**, and denoting the total distance after n training sessions by G_n, write down an expression for G_n in terms of n.

Hence show that $\sum_{n=1}^{N} G_n$ may be expressed in the form $aG_N + bN$, where a and b are integers to be determined. [4]

8 **(i)** It is given that

$$y=\frac{-x^2+4x-5}{x-2},\ x\in\mathbb{R},\ x\neq 2\,.$$

Using an algebraic method, find the set of values that y can take. [3]

(ii) Show that the equation $y=\dfrac{-x^2+4x-5}{x-2}$ can be written as $y=A(x-2)+\dfrac{B}{x-2}$, where A and B are constants to be found. Hence state a sequence of transformations which transform the graph of $y=x+\dfrac{1}{x}$ to the graph of $y=\dfrac{-x^2+4x-5}{x-2}$. [4]

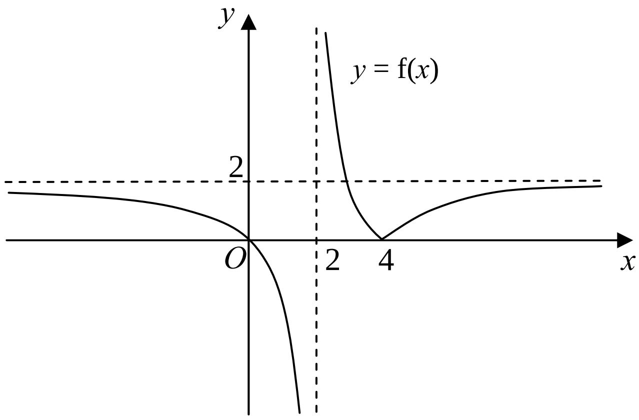
(iii) Sketch the graph of $y=\dfrac{-x^2+4|x|-5}{|x|-2}$, stating the equations of any asymptotes and the coordinates of the turning points. [3]

9 The function f is defined by

$$\mathrm{f}(x)=\begin{cases}\dfrac{2x}{x-2} & \text{for } x<2,\\[2ex] \left|\dfrac{2(4-x)}{x-2}\right| & \text{for } x>2.\end{cases}$$

The graph of $y=\mathrm{f}(x)$ passes through the origin and has an axial intercept at (4, 0).

The lines $x=2$ and $y=2$ are asymptotes to the graph, as shown in the diagram below.



(i) The domain of f is restricted to $x>a$. State the smallest possible value of a such that f^{-1} exists. [1]

Using the value of a found in **(i)**,

(ii) find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}; [4]

Another function g is defined by

$$\mathrm{g}:x\mapsto x^2-6x+7,\qquad x\in\mathbb{R},\ x<3.$$

(iii) Determine whether the composite functions fg and gf exist, justifying your answer. Give a definition (including the domain) of the composite function(s) that exist, and find its range. [Note: There is no need to simplify the rule of the composite function.] [5]

[Turn over

10 Referred to the origin O, the points A and B have position vectors **a** and **b** respectively. It is given that **a** and **b** are perpendicular to each other and have the same magnitude of 3 units each.

Given that A, B and C are collinear,

(i) show that **c** can be expressed as $\mathbf{c} = k\mathbf{b} + (1-k)\mathbf{a}$, where k is a constant. [1]

(ii) Find $|\mathbf{a} \times \mathbf{c}|$, in terms of k, and state its geometrical meaning. [4]

(iii) It is given that the area of triangle OAC is three times the area of triangle OAB.

Find the two possible values of k.

Given also that the length of projection of OC onto OA is 12 units, find **c** in terms of **a** and **b**.

[5]

11 **(a)** Given that the complex number $z = (1+\mathrm{i})t + \dfrac{1-\mathrm{i}}{t}$ is represented by the point P on an Argand diagram where t is a non-zero real constant. Find the Cartesian equation of the locus of the point P. [3]

(b) A fixed complex number a is such that $0 < \arg(a) < \dfrac{\pi}{2}$. On a single Argand diagram, sketch the loci given by $|z-a| = |z-5a|$ and $|z-3a| = 2|a|$. [3]

The two complex numbers that satisfy the above equations are represented by the complex numbers p and q.

If $\arg(p) > \arg(q)$, find the value of $\arg\left(\dfrac{p}{q}\right)$. [3]

Find $|p+q|$ in terms of a. [1]

12 A curve C has parametric equations

$$x = t^2, \quad y = t^3 - 4, \quad t \le 0.$$

(i) The point P on the curve has parameter p. Find the equation of the tangent to C at P. [3]

(ii) Find the exact coordinates of P where the tangent passes through the origin. [3]

(iii) Sketch the curve C. [2]

(iv) Find the exact area of the region bounded by C, the tangent passing through the origin and the y-axis. [4]