2014 H2 Mathematics Prelim P1 Worked Solutions

Qns	Solution	Notes
1	Let the amount of money for a blue voucher, a yellow voucher and a red voucher be *x, y* and *z* respectively. Then $3x+4y+7z=27.40 \quad \cdots\cdots(1)$ $5x+2y+4z=20.80 \quad \cdots\cdots(2)$ $2x+8y+5z=45.00 \quad \cdots\cdots(3)$ From G.C., $x=2, \quad y=5, \quad z=0.20$ $4x+py+2z=43.40$ $p=\dfrac{43.40-4(2)-2(0.2)}{5}=7$	
2 (i)	Let P_n be the statement $u_n=\frac{1}{3}\left[1+8(-2)^n\right]$ for all $n\in\mathbb{Z}_0^+$. LHS of P_0 = $u_0=3$ (Given) RHS of $P_0=\frac{1}{3}\left[1+8(-2)^0\right]=3$ $\therefore$ P_0 is true.	
	Assume that P_k is true for some $k\in\mathbb{Z}_0^+$, ie $u_k=\frac{1}{3}\left[1+8(-2)^k\right]$. We want to prove P_{k+1} is true, ie $u_{k+1}=\frac{1}{3}\left[1+8(-2)^{k+1}\right]$.	
	LHS of P_{k+1} = $u_{k+1}=1-2u_k$ (Given) $=1-\frac{2}{3}\left[1+8(-2)^k\right]$ $=\frac{1}{3}-\frac{16}{3}(-2)^k$ $=\frac{1}{3}+\frac{8}{3}(-2)(-2)^k$ $=\frac{1}{3}\left[1+8(-2)^{k+1}\right]$ $\therefore P_{k+1}$ is true	
	Since P_0 is true and P_k is true $\Rightarrow P_{k+1}$ is true, by Mathematical Induction, P_n is true for all $n\in\mathbb{Z}_0^+$.	
(ii)	The sequence is **divergent** as $n\to\infty$, $(-2)^n$ does not converge to a finite number	

3	when $r = 0$, $3 = 3b$ $r = -1$, $1 = (4 - a)b$	
	<u>**Alternative Method**</u> $(r + 1)^4 + (r + 1)^2 + 1 \equiv (r^2 + ar + 3)(r^2 + r + b)$ $(r^4 + 4r^3 + 6r^2 + 4r + 1) + (r^2 + 2r + 1) + 1 \equiv$ $(r^2 + ar + 3)(r^2 + r + b)$ Comparing coefficients: r^0: $3 = 3b$ r^3: $4 = 1 + a$	
	$a = 3$, $b = 1$	
	$\frac{1}{r^2+r+1}-\frac{1}{r^2+3r+3}=\frac{(r^2+3r+3)-(r^2+r+1)}{(r^2+r+1)(r^2+3r+3)}$ $=\frac{2(r+1)}{(r+1)^4+(r+1)^2+1}$	
	$\sum_{r=0}^{N}\frac{r+1}{(r+1)^4+(r+1)^2+1}$ $=\frac{1}{2}\sum_{r=0}^{N}\left(\frac{1}{r^2+r+1}-\frac{1}{r^2+3r+3}\right)$ $=\frac{1}{2}\left[\begin{array}{c}\frac{1}{1}-\cancel{\frac{1}{3}}\\ +\cancel{\frac{1}{3}}-\cancel{\frac{1}{7}}\\ \vdots\\ +\cancel{\frac{1}{(N-1)^2+(N-1)+1}}-\cancel{\frac{1}{(N-1)^2+3(N-1)+3}}\\ +\cancel{\frac{1}{N^2+N+1}}-\frac{1}{N^2+3N+3}\end{array}\right]$ $=\frac{1}{2}\left(1-\frac{1}{N^2+3N+3}\right)$	
	$\sum_{r=2}^{N}\frac{r}{r^4+r^2+1} = \sum_{r=1}^{N-1}\frac{r+1}{(r+1)^4+(r+1)^2+1}$ $=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{(N-1)^2+3(N-1)+3}\right)$ $=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{N^2+N+1}\right)$	

4 (i)	Area of region R $=5\times\sqrt{3}\ -\ \int_0^{\sqrt{3}}\left(\dfrac{2}{\sqrt{4-x^2}}+3\right)\ \mathrm{d}x$ $=1.37$ (to 3 s.f.)	
(ii)	Equation of new curve $y=\dfrac{2}{\sqrt{4-x^2}}+3-5$ $y=\dfrac{2}{\sqrt{4-x^2}}-2$	
	Volume of revolution $=\pi\int_0^{\sqrt{3}}\left(\dfrac{2}{\sqrt{4-x^2}}-2\right)^2\ \mathrm{d}x$ $=\pi\int_0^{\sqrt{3}}\left(\dfrac{4}{4-x^2}-\dfrac{8}{\sqrt{4-x^2}}+4\right)\mathrm{d}x$ $=4\pi\int_0^{\sqrt{3}}\left(\dfrac{1}{4-x^2}-\dfrac{2}{\sqrt{4-x^2}}+1\right)\mathrm{d}x$ (Shown) $=4\pi\left[\ \dfrac{1}{2(2)}\ln\left\lvert\dfrac{2+x}{2-x}\right\rvert\ -\ 2\sin^{-1}\dfrac{x}{2}\ +\ x\ \right]_0^{\sqrt{3}}$ $=4\pi\left[\ \dfrac{1}{4}\ln\left\lvert\dfrac{2+\sqrt{3}}{2-\sqrt{3}}\right\rvert\ -\ \dfrac{2\pi}{3}\ +\ \sqrt{3}\ \right]$	

5	$\pi rl=k\pi$ $l=\dfrac{k}{r}$ $l^2=r^2+h^2$ $h=\sqrt{\dfrac{k^2}{r^2}-r^2}=\dfrac{\sqrt{k^2-r^4}}{r}$ $V=\dfrac{1}{3}\pi r^2\left(\dfrac{\sqrt{k^2-r^4}}{r}\right)$ $V=\dfrac{\pi r\sqrt{k^2-r^4}}{3}$	
	$\dfrac{\mathrm{d}V}{\mathrm{d}r}=\dfrac{\pi\sqrt{k^2-r^4}}{3}-\dfrac{2\pi r^4}{3\sqrt{k^2-r^4}}$	

	At stationary point, $\frac{dV}{dr}=0$ $\frac{\pi\sqrt{k^2-r^4}}{3}=\frac{2\pi r^4}{3\sqrt{k^2-r^4}}$ $k^2-r^4=2r^4$ $3r^4=k^2$ $r=\sqrt[4]{\frac{k^2}{3}}$	

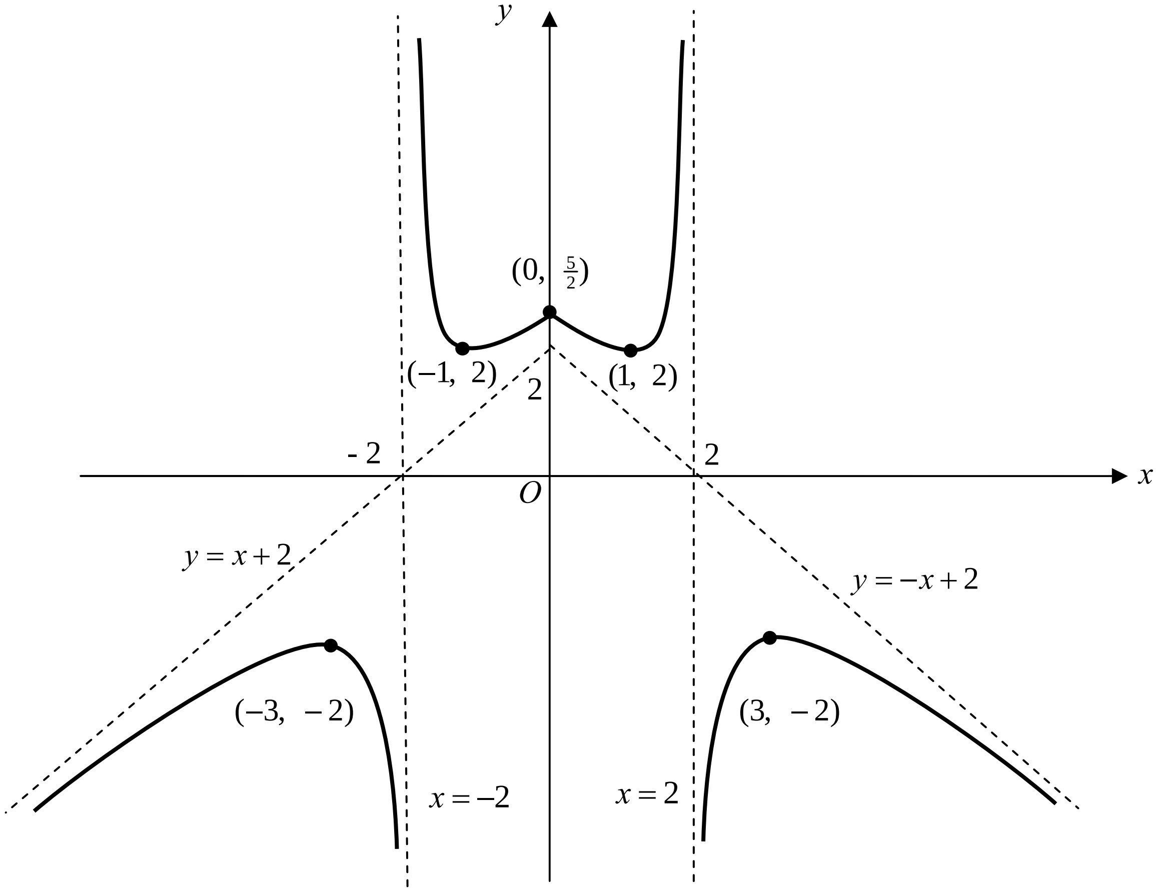
6 (a) (i)	$u=\ln x \qquad \frac{dv}{dx}=\frac{1}{x^2}$ $\frac{du}{dx}=\frac{1}{x} \qquad v=-\frac{1}{x}$	
	$\int_1^n \frac{1}{x^2}\ln x\ dx$ $=\left[-\frac{1}{x}\ln x\right]_1^n - \int_1^n -\frac{1}{x}\left(\frac{1}{x}\right)dx$ $=-\frac{\ln n}{n} - \left[\frac{1}{x}\right]_1^n$ $=-\frac{\ln n}{n} - \left[\frac{1}{n}-1\right]$ $=-\frac{\ln n}{n} - \frac{1}{n}+1$	
(a) (ii)	$\int_1^\infty \frac{1}{x^2}\ln x\ dx = \lim_{n\to\infty}\left[-\frac{\ln n}{n} - \frac{1}{n}+1\right]$ $=1$	
(b)	$x=a\sec\theta \Rightarrow \frac{dx}{d\theta}=a\sec\theta\tan\theta$ When $x=a$, $\sec\theta=1 \Rightarrow \cos\theta=1 \Rightarrow \theta=0$. When $x=2a$, $\sec\theta=2 \Rightarrow \cos\theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}$.	

	$\int_a^{2a} \frac{\sqrt{x^2-a^2}}{x}\ \mathrm{d}x$ $=\int_0^{\frac{\pi}{3}} \frac{\sqrt{a^2\sec^2\theta-a^2}}{a\sec\theta}\ a\sec\theta\tan\theta\ \mathrm{d}\theta$ $=a\int_0^{\frac{\pi}{3}} \tan^2\theta\ \mathrm{d}\theta$ $=a\int_0^{\frac{\pi}{3}} \left(\sec^2\theta-1\right)\ \mathrm{d}\theta$ $=a\left[\tan\theta-\theta\right]_0^{\frac{\pi}{3}}$ $=a\left(\sqrt{3}-\frac{\pi}{3}\right)$	

7	Let A_n denote the distance ran on the nth training session and S_n denote the total distance ran for the n training sessions.	
(i)	$A_n = 7.5 + 0.8(n-1)$ $= 6.7 + 0.8n$	
(ii)	$S_n \geq 475$ $\frac{n}{2}\left[2(7.5)+0.8(n-1)\right]\geq 475$ $7.1n + 0.4n^2 \geq 475$ From GC, $n = 26.7$ Least $n = 27$	
(iii)	For the modified training session, let B_n denote the distance ran on the nth training session and and G_n denote the total distance ran for the n training sessions.	
	$B_6 = 14.93$ $x(1.2)^5 = 14.93$	
	$x = 6$ (nearest integer)	

(iv)	$G_n = \dfrac{6\left[\left(\frac{6}{5}\right)^n - 1\right]}{\frac{6}{5}-1}$ $= 30\left[\left(\frac{6}{5}\right)^n - 1\right]$	
	$\sum_{n=1}^{N} G_n = \sum_{n=1}^{N} 30\left[\left(\frac{6}{5}\right)^n - 1\right]$ $= 30\sum_{n=1}^{N}\left(\frac{6}{5}\right)^n - \sum_{n=1}^{N} 30$ $= \dfrac{30\cdot\frac{6}{5}\left[\left(\frac{6}{5}\right)^N - 1\right]}{\frac{6}{5}-1} - 30N$ $= 6\left\{30\left[\left(\frac{6}{5}\right)^N - 1\right]\right\} - 30N$ $= 6G_N - 30N$	

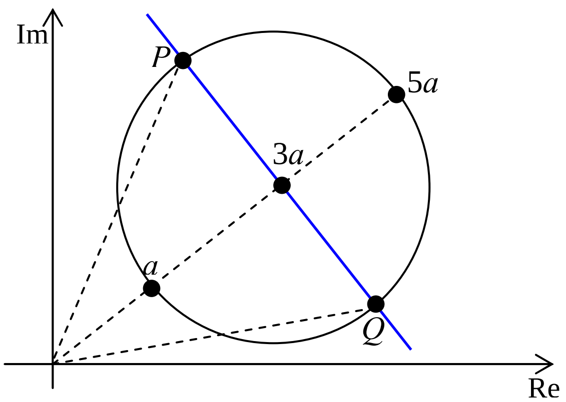
8 **(i)**	$y = \dfrac{-x^2+4x-5}{x-2}$ $\Rightarrow y(x-2) = -x^2+4x-5$ $\Rightarrow x^2+(y-4)x+(5-2y)=0$ For real values of x, discriminant ≥ 0 . $\Rightarrow (y-4)^2 - 4(5-2y) \geq 0$ $\Rightarrow y^2 \geq 4$ $\Rightarrow y \leq -2$ or $y \geq 2$	
(ii)	$\dfrac{-x^2+4x-5}{x-2} = A(x-2) + \dfrac{B}{x-2}$ $\Rightarrow -x^2+4x-5 = A(x-2)^2 + B$ Comparing coefficients of x^2, $A=-1$ Comparing constants, $-5 = 4A+B \Rightarrow B=-1$	
	$y = \dfrac{-x^2+4x-5}{x-2} = -(x-2) - \dfrac{1}{x-2}$ $y = x+\dfrac{1}{x} \xrightarrow{T_1} y = x-2+\dfrac{1}{x-2} \xrightarrow{T_2} y = -\left(x-2+\dfrac{1}{x-2}\right)$	
	The graph of $y = x+\dfrac{1}{x}$ can be transformed to the graph of $y = \dfrac{-x^2+4x-5}{x-2}$ using the following transformations in succession. T_1: Translation of 2 units in the positive x direction. T_2: Reflection in the x-axis.	

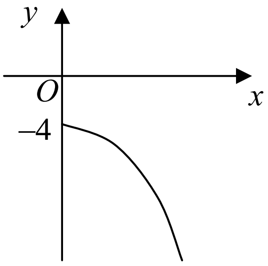
(iii)	

9(i)	least $a = 4$	
(ii)	when $x > 4$, $\quad f(x) = y = \dfrac{2x-8}{x-2}$ $xy - 2y = 2x - 8$ $x(y-2) = 2y - 8$ $x = \dfrac{2y-8}{y-2}$	
	$f^{-1} : x \mapsto \dfrac{2x-8}{x-2}, \quad 0 < x < 2$	
(iii)	$g(x) = x^2 - 6x + 7 = (x-3)^2 - 2, \; x < 3$ $D_g = (-\infty, 3)$ $\qquad D_f = (4, \infty)$ $R_g = (-2, \infty)$ $\qquad R_f = (0, 2)$	
	fg does not exist since $R_g = (-2, -\infty) \not\subset (4, \infty) = D_f$ gf exists since $R_f = (0, 2) \subseteq (-\infty, 3) = D_g$	

	gf(x) = $\mathrm{g}\left(\dfrac{2x-8}{x-2}\right)$ $= \left(\dfrac{2x-8}{x-2}\right)^2 - 6\left(\dfrac{2x-8}{x-2}\right) + 7\,,\ x > 4$	
	From GC: R_{gf} = (–1,7)	
	Alternative method (Mapping method): $(4,\infty) \xrightarrow{f} (0,2) \xrightarrow{g} (-1,7)$ R_{gf} = (–1,7)	

10 **(i)**	Given A, B and C are collinear, $\overrightarrow{AC} = k\overrightarrow{AB}$ $\mathbf{c} - \mathbf{a} = k(\mathbf{b} - \mathbf{a})$ $\mathbf{c} = k\mathbf{b} + (1-k)\mathbf{a}$ (shown)	
(ii)	$\lvert \mathbf{a} \times \mathbf{c} \rvert = \lvert \mathbf{a} \times [k\mathbf{b} + (1-k)\mathbf{a}] \rvert$ $= \lvert k(\mathbf{a} \times \mathbf{b}) + (1-k)(\mathbf{a} \times \mathbf{a}) \rvert$ $= \lvert k \lvert \mathbf{a} \rvert \lvert \mathbf{b} \rvert \sin 90° \hat{\mathbf{n}} + (1-k)\mathbf{0} \rvert$ $= 9\lvert k \rvert$	
	It is the area of a parallelogram with sides OA and OC.	
(iii)	Area of triangle OAC = 3 x area of triangle OAB $\frac{1}{2}\lvert \mathbf{a} \times \mathbf{c} \rvert = \frac{3}{2}\lvert \mathbf{a} \times \mathbf{b} \rvert$ $9\lvert k \rvert = 3\lvert \mathbf{a} \rvert \lvert \mathbf{b} \rvert \sin 90°$ $= 27$ $\lvert k \rvert = 3$ $k = \pm 3$	
(iv)	Length of projection of OC onto OA = 12 $\dfrac{\lvert \mathbf{c} \bullet \mathbf{a} \rvert}{\lvert \mathbf{a} \rvert} = 12$ $\lvert \mathbf{c} \bullet \mathbf{a} \rvert = 12\lvert \mathbf{a} \rvert$ $= 36$	
	When $k = 3$, $\mathbf{c} = 3\mathbf{b} - 2\mathbf{a}$ $\lvert \mathbf{c} \bullet \mathbf{a} \rvert = \lvert (3\mathbf{b} - 2\mathbf{a}) \cdot \mathbf{a} \rvert$ $= \lvert 3(\mathbf{b} \cdot \mathbf{a}) - 2\mathbf{a} \cdot \mathbf{a} \rvert$ $= \lvert 3(0) - 2(3)^2 \rvert$ $= 18$	
	When $k = -3$, $\mathbf{c} = -3\mathbf{b} + 4\mathbf{a}$ $\lvert \mathbf{c} \bullet \mathbf{a} \rvert = \lvert (-3\mathbf{b} + 4\mathbf{a}) \cdot \mathbf{a} \rvert$ $= \lvert -3(\mathbf{b} \cdot \mathbf{a}) + 4\mathbf{a} \cdot \mathbf{a} \rvert$ $= \lvert -3(0) + 4(3)^2 \rvert$ $= 36$	
	$\therefore \mathbf{c} = -3\mathbf{b} + 4\mathbf{a}$	

11 (a)	$z=(1+\mathrm{i})t+\dfrac{1-\mathrm{i}}{t}$ $=\left(t+\dfrac{1}{t}\right)+\left(t-\dfrac{1}{t}\right)\mathrm{i}$ Let $x=t+\dfrac{1}{t}$ ------(1) $y=t-\dfrac{1}{t}$ ------(2) (1) + (2): $x+y=2t$ (1) $-$(2): $x-y=\dfrac{2}{t}$ $\therefore x-y=\dfrac{2}{\frac{x+y}{2}}$ $\Rightarrow x^2-y^2=4$	
11 (b)		
	If $\arg(p)>\arg(q)$, $\arg\left(\dfrac{p}{q}\right)=\arg p-\arg q=\angle POQ$ $=2\tan^{-1}\dfrac{2\|a\|}{3\|a\|}$ $=2\tan^{-1}\dfrac{2}{3}$ or $1.18rad$	
	$\|p+q\|=2\|3a\|=6\|a\|$	

12 (i)	$x=t^2, \quad y=t^3-4$ $\frac{dx}{dt}=2t, \quad \frac{dy}{dt}=3t^2$ $\frac{dy}{dx}=\frac{3t}{2}$	
	Tangent at P $\left(y-p^3+4\right)=\frac{3p}{2}\left(x-p^2\right)$ $2y=3px-p^3-8$	
(ii)	Since the tangent passes through the origin, subst. x = 0 and y = 0 into the equation of tangent in part (i). $-p^3-8=0$ $p=-2$ $x=4, y=-12$ $P(4,-12)$	
(iii)	$x=0, y=-4, t=0$ 	
(iv)	Area $=\frac{1}{2}(4)(12)-\int_{-12}^{-4} x \text{ dy}$ $=24-\int_{-2}^{0}\left(t^2\right)\left(3t^2\right) \text{ dt}$ $=24-\left[\frac{3t^5}{5}\right]_{-2}^{0}$ $=\frac{24}{5}$	