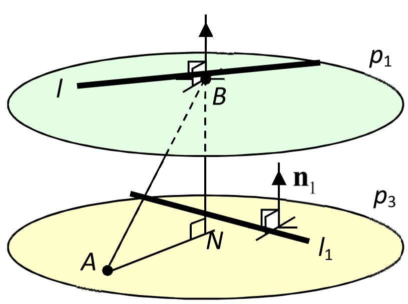
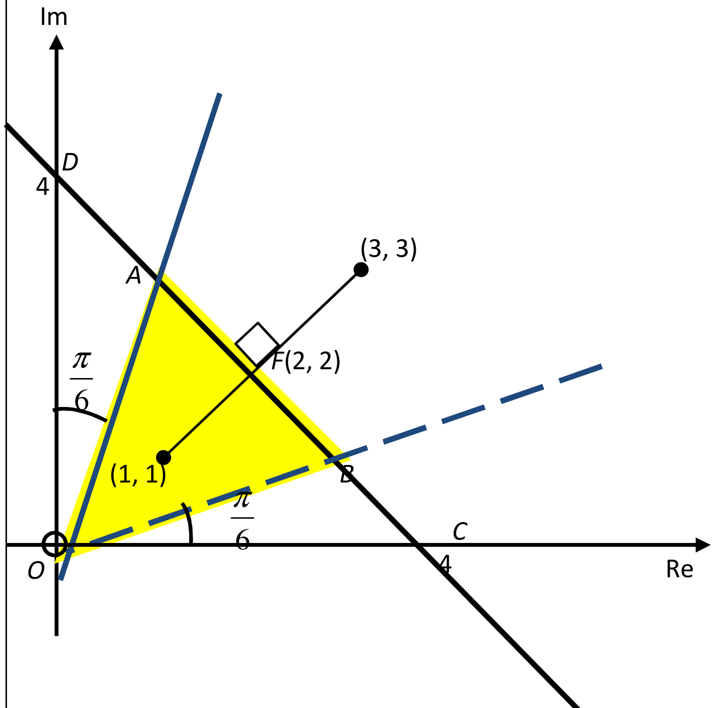
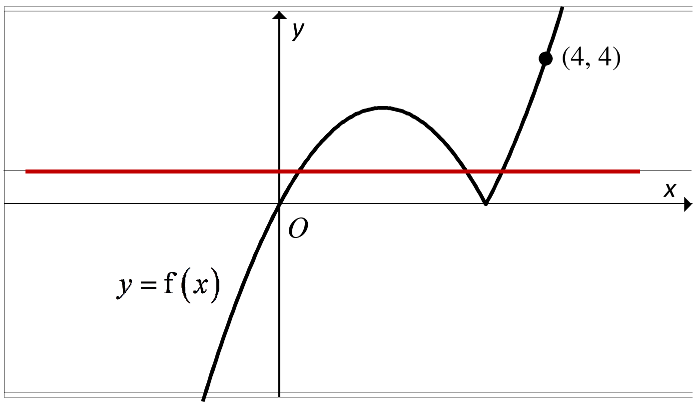
2014 H2 Mathematics C2 Preliminary Examination Solution

Qn	**Solutions**
1(i)	$\frac{1}{\sqrt{k+2x}}=\left(k+2x\right)^{-\frac{1}{2}}$ $=k^{-\frac{1}{2}}\left(1+\frac{2}{k}x\right)^{-\frac{1}{2}}$ $=k^{-\frac{1}{2}}\left[1+\left(-\frac{1}{2}\right)\left(\frac{2}{k}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(\frac{2}{k}x\right)^2+...\right]$ $=k^{-\frac{1}{2}}\left(1-\frac{1}{k}x+\frac{3}{2k^2}x^2+...\right)$ $=k^{-\frac{1}{2}}-k^{-\frac{3}{2}}x+\frac{3}{2}k^{-\frac{5}{2}}x^2+...$
1(ii)	$\left\|\frac{2}{k}x\right\|<1\Rightarrow -1<\frac{2}{k}x<1\Rightarrow -\frac{k}{2}<x<\frac{k}{2}$
1(iii)	Let $x=-\frac{1}{5}$ The value of x is chosen such that $-1<x<1$ and will lead to producing $\sqrt{5}$ and $\sqrt{2}$. $\frac{1}{\sqrt{2-2\left(\frac{1}{5}\right)}}=\frac{1}{\sqrt{\frac{8}{5}}}=\frac{\sqrt{5}}{2\sqrt{2}}$ $\frac{\sqrt{5}}{2\sqrt{2}}\approx\frac{1}{\sqrt{2}}-\left(\frac{1}{2\sqrt{2}}\right)\left(-\frac{1}{5}\right)+\left(\frac{3}{2}\right)\left(\frac{1}{4\sqrt{2}}\right)\left(\frac{1}{25}\right)$ $=\frac{1}{\sqrt{2}}+\frac{1}{10\sqrt{2}}+\frac{3}{200\sqrt{2}}$ $=\frac{223}{200\sqrt{2}}$ $\therefore\sqrt{5}\approx\frac{223}{100}$ <u>NOTE</u>: Another suitable value to be used is $x=\frac{1}{4}$ and using this value, $\sqrt{5}\approx\frac{256}{115}$.

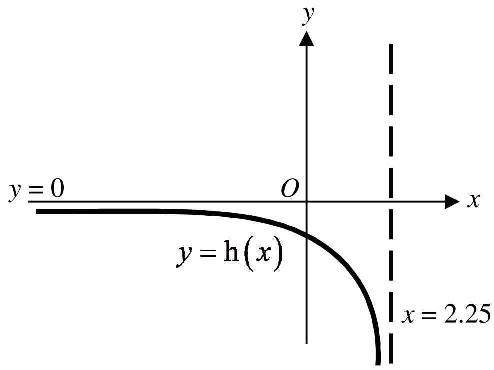
2(i)	$\cos\theta = \dfrac{\left\lvert \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -7 \\ 2 \end{pmatrix} \right\rvert}{\sqrt{14}\sqrt{78}} = \dfrac{\lvert 10-7-6 \rvert}{\sqrt{14}\sqrt{78}} = \dfrac{3}{\sqrt{14}\sqrt{78}}$ $\Rightarrow \theta = 84.8^{\circ}$
2(ii)	From GC, $\ell : r = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \lambda \in \mathbb{R}$
2(iii)	Since ℓ and ℓ_1 are perpendicular $\Rightarrow \ell$ is not parallel to ℓ_1. Note that ℓ_1 is parallel to p_1 and A is not on p_1 $\Rightarrow \ell_1$ is not on p_1. $\therefore$ ℓ and ℓ_1 do not intersect. Hence, ℓ and ℓ_1 are skew lines.
2(iv)	Direction vector of $\ell_1 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}$ $\therefore \ell_1 : r = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}, \ \mu \in \mathbb{R}$
2(v)	<u>Method 1 (Compute length of project vector)</u> Choose $A = (0, 0, -2)$ on p_3 and $B = (-2, -1, 0)$ on p_1.  Required perpendicular distance is $\left\lvert \overrightarrow{AB} \bullet \hat{n} \right\rvert = \dfrac{1}{\sqrt{14}} \left\lvert \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \right\rvert = \dfrac{1}{\sqrt{14}} \lvert -4-1-6 \rvert = \dfrac{11}{\sqrt{14}}$.

	Method 2 (Use of foot of perpendicular) Choose $A=(0,0,-2)$ on p_3. Obtain equation of line through A and perpendicular to $p_1 \Rightarrow \mathbf{r}=\begin{pmatrix}0\\0\\-2\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-3\end{pmatrix}$ Find foot of perpendicular N through intersection of line and p_1 $\Rightarrow \left(\begin{pmatrix}0\\0\\-2\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-3\end{pmatrix}\right)\cdot\begin{pmatrix}2\\1\\-3\end{pmatrix}=-5 \Rightarrow \lambda=\frac{-11}{14}$ Thus $\overrightarrow{NA}=\overrightarrow{ON}-\overrightarrow{OA}=\lambda\begin{pmatrix}2\\1\\-3\end{pmatrix}=\frac{-11}{14}\begin{pmatrix}2\\1\\-3\end{pmatrix}$ and required perpendicular distance is $NA=\frac{11}{14}\left\|\begin{pmatrix}2\\1\\-3\end{pmatrix}\right\|=\frac{11\sqrt{14}}{14}$. Perpendicular distance between ℓ and ℓ_1 equals to the perpendicular distance between the two planes $=\frac{11}{\sqrt{14}}$. (Refer to same diagram above)
3(a)(i)	$z^5=-32$ $z^5=32\mathrm{e}^{\mathrm{i}(\pi)}$ $z^5=2^5\mathrm{e}^{\mathrm{i}(\pi+2k\pi)}$ Therefore $z=2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{5}+\frac{2k\pi}{5}\right)}, \quad k=0,\pm1,\pm2$.
3(a)(ii)	Since the two complex numbers are in the 1st and 2nd quadrants corresponding to $k=0$ and $k=1$, thus $w=\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{5}\right)}$.

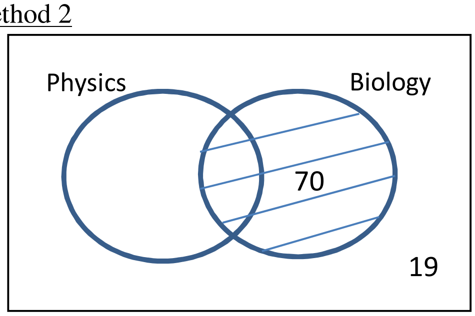
3(b) (i)	
3(b) (ii)	<u>Method 1 (Using ½ x base x height)</u> $OF=\sqrt{2^2+2^2}=\sqrt{8}$ $\tan\angle FOA=\tan\left(\frac{\pi}{12}\right)=\frac{FA}{OF}$ $\Rightarrow FA=\sqrt{8}\tan\left(\frac{\pi}{12}\right)$ Area of triangle $=\frac{1}{2}\times AB\times OF$ $=\frac{1}{2}\left(\sqrt{8}\right)2\sqrt{8}\tan\left(\frac{\pi}{12}\right)=8\tan\left(\frac{\pi}{12}\right)=2.14$ (to 3 s.f.) <u>Method 2 (Using sum of areas of triangles)</u> Let $OA=OB=x$ Area $\Delta OAB+$ area $\Delta OAC+$ area $\Delta OAD=$ area ΔOCD $\Rightarrow \frac{1}{2}x^2\sin\frac{\pi}{6}+2\times\left[\frac{1}{2}(x)(4)\sin\frac{\pi}{6}\right]=\frac{1}{2}(4)^2$ $\Rightarrow x^2+8x-32=0$ $\Rightarrow x=-4\pm4\sqrt{3}$ $\Rightarrow x=-4+4\sqrt{3}$ since $x>0$ Thus, area of shaded region $=\frac{1}{2}x^2\sin\frac{\pi}{6}=4\left(\sqrt{3}-1\right)^2=16-8\sqrt{3}$

3(b)(iii)	Note that the point (5, –1) lies on the perpendicular bisector. Therefore $\frac{3\pi}{4} \le \arg(z-5+\mathrm{i}) < \pi - \tan^{-1}\left(\frac{1}{5}\right)$. If correct to 3 s.f., answer is $2.36 \le \arg(z-5+\mathrm{i}) < 2.94$.
4(i)	<u>Method 1 (Horizontal line test)</u>  From the graph above, a horizontal line drawn cuts the curve $y = \mathrm{f}(x)$ at 3 times. Thus function is not one-to-one and inverse does not exist. <u>Method 2 (Use a counter example)</u> $\mathrm{f}(0) = \mathrm{f}(3) = 0$ $y = \mathrm{f}(x)$ is not one-to-one, therefore f^{-1} does not exist.
4(ii)	Largest value of $m = 1.5$.
4(iii)	Let $y = -x(x-3)$ $y = -\left(x-\frac{3}{2}\right)^2 + \frac{9}{4}$ $x = \frac{3}{2} \pm \sqrt{\frac{9}{4} - y}$ Since $x < 1.5$, $\mathrm{f}^{-1}(x) = \frac{3}{2} - \sqrt{\frac{9}{4} - x}$, $x < 2.25$.
4(iv)	<u>Method 1 (Taking f^{-1} on both sides)</u> From $\mathrm{fg}(x) = \frac{9}{4} - x^2$ $\Rightarrow \mathrm{f}^{-1}\mathrm{fg}(x) = \mathrm{f}^{-1}\left(\frac{9}{4} - x^2\right)$ $\Rightarrow \mathrm{g}(x) = \frac{3}{2} - \sqrt{x^2}$ Since $x < 0$, $\mathrm{g}(x) = \frac{3}{2} + x$.

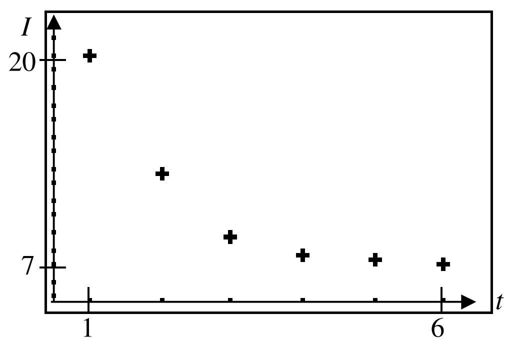
	Method 2 (Making use of composition of functions) From $\mathrm{fg}(x)=\frac{9}{4}-x^2$ $\Rightarrow \mathrm{g}(x)\left(3-\mathrm{g}(x)\right)=\frac{9}{4}-x^2$ $\Rightarrow \left(\mathrm{g}(x)\right)^2-3\mathrm{g}(x)+\frac{9}{4}-x^2=0$ Treating the equation as a quadratic equation in $\mathrm{g}(x)$. $\Rightarrow \mathrm{g}(x)=\dfrac{3\pm\sqrt{9-4\left(\frac{9}{4}-x^2\right)}}{2}$ $\Rightarrow \mathrm{g}(x)=\dfrac{3\pm\sqrt{4x^2}}{2}=\dfrac{3\pm(-2x)}{2}$ since $x<0$. As domain of f is $x<1.5$, thus $\mathrm{g}(x)=\dfrac{3+2x}{2}$. Method 3 (By comparison of functions) Since $\mathrm{f}\left(\mathrm{f}^{-1}(x)\right)=x$ $\Rightarrow \mathrm{f}\left(\mathrm{f}^{-1}\left(\frac{9}{4}-x^2\right)\right)=\frac{9}{4}-x^2$ $\Rightarrow \mathrm{f}^{-1}\left(\frac{9}{4}-x^2\right)=\mathrm{g}(x)$ $\Rightarrow \mathrm{g}(x)=\frac{3}{2}-\sqrt{x^2}$ Since $x<0$, $\mathrm{g}(x)=\frac{3}{2}+x$.
4(v) (a)	$y=\dfrac{3}{x-a}$ $\Big\downarrow$ Translate 10 units in positive y-direction. $y=10+\dfrac{3}{x-a}$ $\Big\downarrow$ Stretch with scale factor $\frac{1}{2}$ parallel to x-axis. ($x\to 2x$) $y=10+\dfrac{3}{2x-a}=\mathrm{k}(x)$
3(v) (b)	For hf to exist, $R_{\mathrm{f}}\subseteq D_{\mathrm{h}}$ Since $R_{\mathrm{f}}=(-\infty, 2.25)$, therefore $D_{\mathrm{h}}=(-\infty, a)$.

	Thus the smallest value of a is 2.25.  $\therefore D_{\mathrm{f}} \xrightarrow{\mathrm{f}} R_{\mathrm{f}} \xrightarrow{\mathrm{h}} (-\infty, 0) \Rightarrow R_{\mathrm{hf}} = (-\infty, 0)$

Section B: Statistics

Qn	**Solution**
5	No. of ways that all the girls are seated together $= 6! \times 5! = 86400$ No. of ways that not all the girls are seated together $= 10! - 86400 = 3542400$
	No. of ways for the two particular girls to be seated directly opposite each other $= 8! = 40320$ No. of ways that two particular girls are seated directly opposite each other and 4 of the boys are seated together $= {}^5C_4 \times 4! \times 4! \times 2$ $= 5760$ Probability $= \dfrac{5760}{40320} = \dfrac{1}{7}$
6(i)	<u>Method 1</u> No. of students who study Physics and Biology $= 32 + 70 - (100 - 19) = 21$ No. of students who study Physics only $= 32 - 21 = 11$ Probability $= \dfrac{11}{100} = 0.11$ <u>Method 2</u>  No. of students who study Physics only $= 100 - 70 - 19 = 11$ Probability $= \dfrac{11}{100} = 0.11$

(ii)	No. of male students who study both Physics and Biology $= 24+33-(50-8)=15$ Probability = $\dfrac{15}{50}=0.3$
(iii)	<u>Method 1</u> $\mathrm{P}(X\cap Y)=\dfrac{8}{100}$ $\mathrm{P}(X)\times\mathrm{P}(Y)=\dfrac{1}{2}\times\dfrac{32}{100}=\dfrac{16}{100}\neq\dfrac{8}{100}$ Therefore the two events are not independent. <u>Method 2</u> $\mathrm{P}(X\mid Y)=\dfrac{8}{32}=\dfrac{1}{4}\neq\dfrac{1}{2}=\mathrm{P}(X)$ Therefore the two events are not independent
7(i)	Let S denote the speed of a randomly chosen passenger car. Let $X=S_1+S_2+S_3-2S$. $X\sim\mathrm{N}(3\times85-2\times85,3\times20^2+2^2 20^2)$ $\quad=\mathrm{N}(85,2800)$ $\mathrm{P}(\lvert S_1+S_2+S_3-2S\rvert\le 50)$ $=\mathrm{P}(-50\le X\le 50)$ $=0.249\quad(3\text{ s.f.})$
(ii)	$\mathrm{P}\left(\dfrac{5}{60}S_1+\dfrac{5}{60}S_2+\ldots+\dfrac{5}{60}S_{80}\le 550\right)$ Let $T=\dfrac{5}{60}S_1+\dfrac{5}{60}S_2+\ldots+\dfrac{5}{60}S_{80}$. $T\sim\mathrm{N}\left(\dfrac{5}{60}(80\times85),\left(\dfrac{5}{60}\right)^2(80\times20^2)\right)$ $T\sim\mathrm{N}\left(\dfrac{1700}{3},\dfrac{2000}{9}\right)$ $\mathrm{P}(T\le 550)=0.132\quad(3\text{ s.f.})$ **Method 2:** $\mathrm{P}\left(\dfrac{5}{60}S_1+\dfrac{5}{60}S_2+\ldots+\dfrac{5}{60}S_{80}\le 550\right)$ $=\mathrm{P}(S_1+S_2+\ldots+S_{80}\le 6600)$ Let $T=S_1+S_2+\ldots+S_{80}$. $T\sim\mathrm{N}(80\times85,80\times20^2)=\mathrm{N}(6800,32000)$ $\mathrm{P}(T\le 6600)=0.132\quad(3\text{ s.f.})$ Assume that the cars are travelling independently.

8(i)	$\mathrm{P}(S<115)=0.95 \Rightarrow \mathrm{P}\left(Z<\frac{115-\mu}{\sigma}\right)=0.95$ $\mathrm{P}(S>80)=0.75 \Rightarrow \mathrm{P}\left(Z<\frac{80-\mu}{\sigma}\right)=0.25$ $\begin{cases}\frac{115-\mu}{\sigma}=1.6448536---(1)\\ \frac{80-\mu}{\sigma}=-0.6744897---(2)\end{cases}$ $35=2.319343\sigma$ $\sigma=15.1$ (3 s.f.) $\mu=90.2$ (3 s.f.)
(ii)	Let Y denote the no. of pupils with height > 115 cm. $Y \sim \mathrm{B}(60, 0.05)$ Since $n=60>50$ and $np=3<5$, $\therefore Y \sim \mathrm{Po}(3)$ approx $\mathrm{P}(Y<3)$ $=\mathrm{P}(Y\le 2)=0.423$
(iii)	Quota sampling. The university student can set a quota of 30 kindergarten boys and 30 kindergarten girls to be sampled. She is free to choose the boys and girls to fulfill her quota.
9(i)	
(ii)	Since the scatter diagram shows a curvilinear relationship, Model (A) will not be suitable. The scatter diagram shows a decreasing concave upwards trend. Hence Model (B) is not suitable as it has a decreasing concave downwards shape. Model (C) will be the most suitable model as it has a decreasing concave upwards shape.
(iii)	$I=a\mathrm{e}^{bt}$ $\Rightarrow \ln I=\ln a+bt$ From GC, the regression line is $\ln I=2.980120727-0.2013264777t$ Hence $\ln a=2.980120727$ $a=19.7$

	$b = -0.201$ The value of a is the initial radiation intensity of the radioactive source.
(iv)	$\ln(8.0) = 2.980120727 - 0.2013264777t$ $t = 4.47$ Since the value of I is within the data range and $\lvert r \rvert = 0.911$ which is close to 1, the prediction will be reliable.
10(i)	$s^2 = \frac{1}{8}\left(\sum (x-4)^2 - \frac{\left(\sum (x-4)\right)^2}{9}\right)$ $= \frac{1}{8}\left(7.38 - \frac{8^2}{9}\right) = 0.0336$
(ii)	$H_0 : \mu = 5$ against $H_1 : \mu < 5$ Using t-test, test statistics $t = \dfrac{\overline{X} - \mu}{\frac{s}{\sqrt{n}}}$ Since p-value = $0.0532 > 0.05$, hence H_0 is not rejected at the 5% level of significance, i.e. there is insufficient evidence to claim that the company overstated the mean mass of coffee in a capsule. The assumption is that X, the mass of coffee in one capsule, is normally distributed.
(iii)	$H_0 : \mu = 5$ $H_1 : \mu \neq 5$
(iv)	$-1.959963986 < \dfrac{\overline{x} - 5}{\frac{0.25}{\sqrt{100}}} < 1.959963986$ $4.95 < \overline{x} < 5.05$
(v)	$4.94 < \overline{x} < 5.06 \Rightarrow$ $-2.4 < \dfrac{\overline{x} - 5}{\frac{0.25}{\sqrt{100}}} < 2.4$ Probability = $1 - P(-2.4 < Z < 2.4)$ $= 0.0164$
11(i)	Let X denote the number of customers wanting to rent a camera. $X \sim \text{Po}(2)$ $P(X = 0) = 0.135335 = 0.135$ (3 s.f.)
(ii)	$P(X > 3) = 1 - P(X \leq 3)$ $= 0.1428765 = 0.143$
(iii)	Let Y denote the number of days out of 7 days where no cameras are rented. $Y \sim B(7, 0.1353352832)$ $P(Y \leq 3) = 0.9916589 = 0.992$

(iv)	Required probability $=\left[\mathrm{P}(X=0)\right]^2\left[\mathrm{P}(X>3)\right]\dfrac{3!}{2!}$ $=0.00785$ (3 s.f.)
(v)	Let W denote the number of customers renting a camera in a week. $W \sim \mathrm{Po}(14)$ Since $\lambda = 14 > 10, \therefore W \sim \mathrm{N}(14,14)$ approx $\mathrm{P}(W \le 8) \underset{cc}{\rightarrow} \mathrm{P}(W < 8.5) = 0.0708$
(vi)	Let A denote the number of days in a week with no rental of camera. A ~ B(7, 0.135335) Mean = 0.94735, variance = 0.819137 As n = 52 is large, by CLT $\overline{A} \sim \mathrm{N}\left(0.94735, \dfrac{0.819137}{52}\right)$ approx $\mathrm{P}\left(\overline{A} \le 1\right) = 0.663$ (3 s.f.)