

Section A: Pure Mathematics [40 marks]

- 1 (i) Given that $y = (\cos^{-1} x)^2$, show that

$$(1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} = 2. \quad [3]$$

- (ii) Hence find the Maclaurin's series for $(\cos^{-1} x)^2$, up to and including the term in x^3 .
Give the coefficients in exact form. [3]

- (iii) Show that, if x is sufficiently small for powers of x above x^2 to be neglected, then
 $e^{(\cos^{-1} x)^2} \approx e^{\frac{\pi^2}{4}} (1+ax+bx^2)$, where a and b are constants to be determined exactly.
[You may use standard results given in the List of Formulae (MF15).] [3]

- 2 A family of curves is given by the differential equation

$$x \frac{dy}{dx} - y = x(x-y).$$

By substituting $y = vx$, show that the family of curves is given by

$$y = x(1 - Ae^{-x}). \quad [6]$$

Sketch, on a single diagram, three members of the family of curves corresponding to positive values of A . [2]

Find a cartesian equation of the locus of the stationary points for the family of curves corresponding to positive values of A . [1]

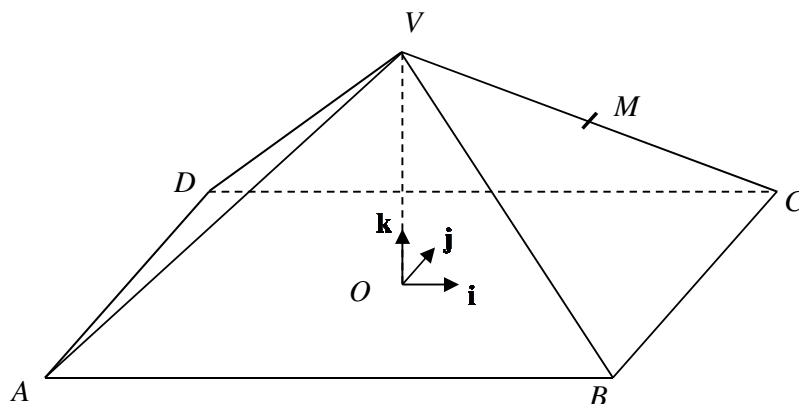
- 3 The function f is defined by

$$f : x \mapsto \frac{5}{(x-4)^2} \text{ for } x \in \mathbb{R}, x < 4.$$

- (i) Find $f^{-1}(x)$ and state the domain of f^{-1} . [3]
- (ii) On a single clearly labelled diagram, sketch the graphs of $y = f(x)$, $y = f^{-1}(x)$ and $y = ff^{-1}(x)$. [4]
- (iii) Show that the x -coordinates of the points of intersection of the curves $y = f(x)$ and $y = f^{-1}(x)$ satisfy the equation

$$x^3 - 8x^2 + 16x - 5 = 0.$$

Hence find the values of these x -coordinates in exact form. [3]



In the diagram, O is centre of the rectangular base $ABCD$ of a right pyramid with vertex V . Perpendicular unit vectors \mathbf{i} , \mathbf{j} , \mathbf{k} are parallel to AB , BC , OV respectively. The length of AB , BC and OV are 12 units, 6 units and 6 units respectively. The point M is the mid-point of CV and the point O is taken as the origin for position vectors.

- (i) Show that the equation of the line AM may be expressed as $\mathbf{r} = \begin{pmatrix} -6 \\ -3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$, where t is a parameter. [3]
- (ii) Find the perpendicular distance from B to the line AM . [3]
- (iii) Find the acute angle between the line DV and the plane AMB . [4]

The plane Π has equation $\mathbf{r} \cdot \begin{pmatrix} -1 \\ 4 \\ a \end{pmatrix} = 4$.

- (iv) Given that the three planes AMB , AMD and Π have no point in common, find the value of a . [2]

Section B: Statistics [60 marks]

- 5** An automatic dispensing machine is set to dispense hot chocolate into cups. Based on observations over a long period of time, the cups were found to contain volumes of hot chocolate, in ml, with mean 55 and standard deviation 10.

A large random sample of n cups of hot chocolate is taken. Given that the probability that the sample mean exceeds 54 ml is more than 0.77, find the least value of n . [3]

- 6** The Ministry of Urban Development wishes to find out how the recent property cooling measures have affected Singaporeans. A survey is to be carried out on 5% of the households in a particular constituency.

- (i) Explain how a systematic sample might be carried out. [2]
- (ii) Describe an alternative sampling method which would be better in this case. [2]

- 7** A blogger observes that his blog receives an average of 3.2 visitors a day. Assuming a Poisson distribution,

- (i) find the probability that his blog receives at least 5 but fewer than 10 visitors over a two-day period, [2]
- (ii) use a suitable approximation to find the probability that his blog receives more than 30 visitors in a particular week. [3]

Explain why the Poisson distribution may not be a good model for the number of visitors his blog receives in a year. [2]

- 8** A group of seven people consisting of two single men, three single women and one married couple are arranged to be seated in a row.

- (i) Find the probability that the married couple is seated together. [2]
- (ii) The married couple is seated together. Find the probability that all the women are seated next to one another. [2]
- (iii) Find the probability that the men and women are seated alternately. [2]

The seven people are now seated at a round table.

- (iv) Find the probability that one particular woman is seated between two men. [3]

- 9 In a standardized assessment test conducted annually, it is found that, on average, 1 in 5 students who sit for the test are awarded distinction.

- (i) Out of 23 students who sit for the test, find the expected number of students who will be awarded distinction. [1]
- (ii) Find the minimum number of students who must sit for the test so that the probability of having at least 9 students being awarded distinction is more than 0.7. [3]

It is further found that, on average, 93% of students who sit for the test pass the test.

- (iii) 60 students from a junior college sit for the test. Use a suitable approximation to find the probability that more than 55 students pass the test. [3]

- 10 'Famous Amas' makes two types of cookies, almond cookie and chocolate chip cookie. The mass of an almond cookie and a chocolate chip cookie, in grams, are independent random variables with the distribution $N(32, 3.0^2)$ and $N(28, 1.8^2)$ respectively.

- (i) Find the probability that the total mass of four randomly chosen chocolate chip cookies is between 100 g and 120 g. [2]
- (ii) Find the probability that total mass of four randomly chosen chocolate chip cookies differs from twice the mass of a randomly chosen almond cookie by less than 50 g. [3]

Almond cookies are sold at \$6 per 100 g and chocolate chip cookies at \$5 per 100 g. John buys 4 almond cookies and 4 chocolate chip cookies. Cindy buys 10 chocolate chip cookies.

- (iii) Find the probability that John pays more than Cindy. [5]

- 11 The Department of Environmental Health did a research on how the number of mosquitoes, x , in a container varies with time, t . Observations on successive days give the data shown in the following table.

t	1	2	3	4	5	6	7
x	32	47	65	92	132	190	275

- (i) Draw a scatter diagram for the data. Comment on whether a linear model would be appropriate. [3]
- (ii) It is given that x can be modelled by $x = ae^{bt}$, where a and b are constants. Find the equation of the least squares regression line of $\ln x$ on t . Hence calculate the estimates of a and b . [4]
- (iii) Calculate an estimate for the number of mosquitoes in the container if the experiment has lasted for 30 days. [1]
- (iv) Comment on the reliability of your answer in part (iii). [1]

- 12** A manufacturer claims that the mean lifetime of the light bulbs he produces is at least 1200 hours. A random sample of 120 bulbs is taken and the lifetime per bulb, x hours, is measured. The results are summarised by

$$\sum(x-1200) = -60 \quad \text{and} \quad \sum(x-1200)^2 = 2014.$$

- (i) Find unbiased estimates of the population mean and variance. [2]
- (ii) Test the manufacturer's claim at the 10% significance level. [4]

The same manufacturer makes a change in the process which is intended to improve the average lifetime. A random sample of 20 bulbs is taken from the output of the new process. The mean and standard deviation of this sample are k hours and 9.8 hours respectively. A test at the 5% significance level indicates that the manufacturer's claim is valid for this improved process.

- (iii) Stating a necessary assumption, find the least possible value of k , giving your answer correct to 3 decimal places. [5]