1. Three neighbours, Jasmine, Joyce and Jolin bought three different kinds of prawns in the market. They were unable to recall the individual prices per kilogram for each type of prawn but they can remember the total amount that they each paid. The weights of the prawns and the total amount paid are shown in the following table.

	Jasmine	Joyce	Jolin
Tiger Prawn (kg)	1.2	1.0	2.0
Brown Prawn (kg)	2.0	1.0	2.0
King Prawn (kg)	1.5	2.0	1.0
Total amount paid ($)	76.53	71.50	72.50

Assuming that there is no change in the price per kilogram paid by each of the neighbours, find the price per kilogram for each type of prawn. [3]

2. By using the substitution $u = 1 - x$, show that $\int_0^1 x^n (1-x)^m \, dx = \int_0^1 (1-x)^n x^m \, dx$. [2]

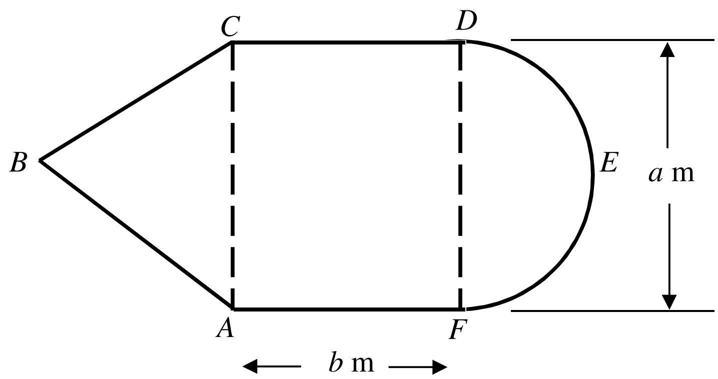
Hence, or otherwise, evaluate $\int_0^1 x^2 \sqrt{1-x} \, dx$, express your answer in exact form. [3]

3. One of the roots of the equation $z^3 - 2z^2 + az + 1 + 3i = 0$ is $z = i$. Find the complex number a and the other roots. [5]

4. Solve the inequality $\dfrac{x+6}{x-2} \geq x+1$. [3]

Hence solve the inequality $\dfrac{1+6x}{1-2x} \geq \dfrac{x+1}{x}$. [3]

5. The diagram shows a field consisting of an equilateral triangle ABC, a rectangle $ACDF$ and a semi-circle DEF.



Suppose $DF = a$ m, $AF = b$ m, and the area of the field is fixed at 400 m^2.
Find, using differentiation, the values of a and b which give a field of minimum perimeter, giving your answers correct to 2 decimal places. [8]

[Turn over

6. The area of the island Singadives is $3000\,\text{km}^2$ at the end of 2014. Due to the rise in sea level, the area of the island decreases gradually every year. At the end of 2015, the area decrement is $60\,\text{km}^2$. Two companies, A and B, are engaged to study the trend in the area decrement of Singadives.

(a) According to Company A, the area decrement in each subsequent year is $5\,\text{km}^2$ less than that of the previous year. Find the area of Singadives at the end of 2023. [2]

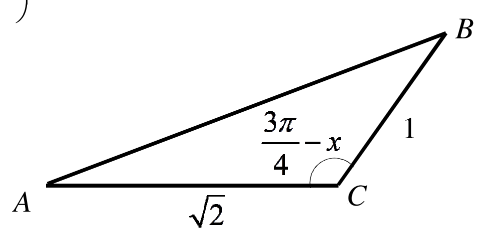
(b) According to Company B, the area decrement in each subsequent year is $\frac{5}{6}$ that of the previous year. Let 2015 be the first year.

(i) Find the total area decrement in the first n years. [2]

(ii) At the end of which year will the area of Singadives first fall below $2720\,\text{km}^2$? [3]

(iii) Hence explain clearly why the area of Singadives will always be greater than $2640\,\text{km}^2$. [2]

7. **(a)** In a triangle ABC as shown in the diagram below, $AC=\sqrt{2}$, $BC=1$ and angle $ACB=\left(\frac{3\pi}{4}-x\right)$ radians.



Show that $AB^2=3+2(\cos x-\sin x)$. [3]

It is given that x is sufficiently small for x^4 and higher powers of x to be neglected. By using the standard series given in the list of formulae (MF15), find the series expansion of AB^2 up to and including the term in x^3. [2]

(b) Given that $\cos y = hx+kx^2$ where h and k are constants, find the Maclaurin series for y up to and including the term in x^2. [3]

If $\cos(y+2)=hx+kx^2$, deduce the Maclaurin series for y up to and including the term in x^2. [1]

8. Relative to the origin O, the points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}+\mathbf{b}$ respectively. The point X is on AB produced such that $AB:AX$ is $1:5$ and the point Y is such that $ACXY$ is a parallelogram. Given that the area of the triangle OAB is 1 square unit and $\mathbf{b}$ is a unit vector.

(i) Find in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vectors of X and Y. Hence show that $OABY$ is a trapezium. [5]

(ii) Give a geometrical meaning of $\left|(\mathbf{a}+\mathbf{b})\cdot\mathbf{b}\right|$. [1]

(iii) Find the area of $ACXY$. Hence find the shortest distance from X to the line that passes through the points A and C. [3]

9. A curve C has parametric equations

$$x=\sin t, \qquad y=\sin 2t, \qquad \text{where } -\frac{\pi}{3}\le t\le 0.$$

(i) Sketch C, showing clearly the coordinates of the end points. [2]

(ii) Find the equations of the tangent and the normal to the curve at the point P where $t=-\frac{\pi}{6}$. [5]

(iii) The tangent and the normal at P meet the y-axis at the points Q and R respectively. Find the area of the quadrilateral $PRQS$ where S is the foot of the perpendicular from P to the x-axis. [3]

10. A sequence $\{u_n\}$ is such that $u_1=\frac{1}{2}$ and $u_{n+1}=u_n+\frac{1-n}{2^{n+1}}$ for all $n\ge 1$.

(i) Use the method of mathematical induction to prove that $u_n=\frac{n}{2^n}, n\ge 1$. [4]

(ii) Determine if the sequence $\{u_n\}$ converges. [1]

(iii) Using the method of differences, find $\sum_{n=1}^{N}\frac{1-n}{2^{n+1}}$. [2]

(iv) Hence find $\sum_{n=1}^{N}u_n$. [3]

11. A curve has equation given by $y=(\ln x)^2-1$, where $x>0$.

(i) Sketch the graph, indicating the exact coordinates of the x-intercepts and the turning point. [4]

The region R is bounded by the curve and the x-axis.

(ii) Find the exact area of R. [4]

(iii) Find the volume of the solid generated when R is rotated through 2π radians about the y-axis. [4]

12. A botanist discovers that a particular species of bamboo exhibits different growth behaviours in countries A and B. The length of the leaf of this particular species of bamboo t days after sprouting is x cm.

(a) In country A, it is found that the growth rate of the bamboo leaf is modelled by the differential equation

$$t\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\frac{\mathrm{d}x}{\mathrm{d}t}=4t-1.$$

It is observed that the length of the bamboo leaf one day after sprouting is 2 cm.

(i) Differentiate $t\frac{\mathrm{d}x}{\mathrm{d}t}$ with respect to t. Hence show that the general solution of the differential equation is given by

$$x=t^2-t+C\ln t+2,$$

where C is an arbitrary constant. [4]

(ii) Sketch two typical members of the family of solution curves. [2]

(b) In country B, the length of the bamboo leaf increases at a rate proportional to its length, and at the same time, due to the loss of water in the leaf, the length of the leaf decreases at a rate proportional to the square of its length. It is noted that the length of the bamboo leaf when it first sprouts is 1 cm and when the length of the bamboo leaf reaches 6 cm, it remains at this constant value.

(i) Show that the growth rate of the bamboo leaf in country B can be modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t}=kx(6-x), \text{ where } k \text{ is a positive constant.}$$ [2]

(ii) Solve the differential equation in part **(b)(i)**, expressing x in terms of k and t. [4]

(iii) Given that the bamboo leaf experiences the greatest growth rate 6 days after sprouting, find the value of k, leaving your answer correct to 5 decimal places. [2]