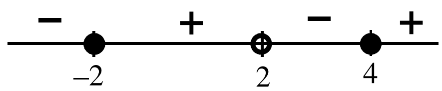
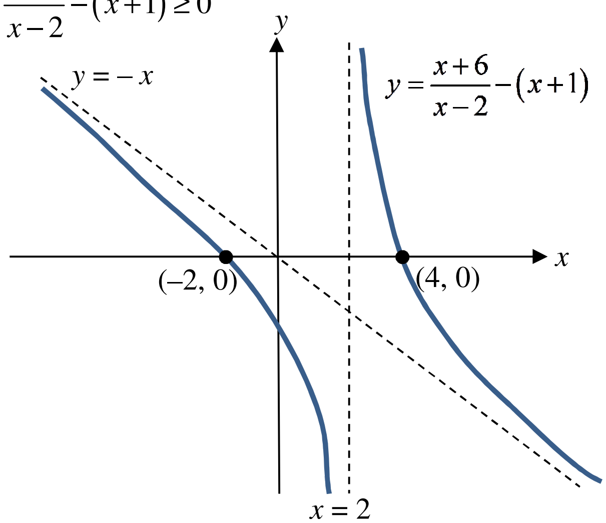
2014 H2 Mathematics C2 Prelims Paper 1 Solutions

Qn	Solutions
1.	Let \$ x be the price of tiger prawn per kg, let \$ y be the price of brown prawn per kg and let \$ z be the price of king prawn per kg. $1.2x+2y+1.5z=76.53 \quad ---- \ (1)$ $x+y+2z=71.50 \quad ---- \ (2)$ $2x+2y+z=72.50 \quad ---- \ (3)$ From GC: $x=9.65, \quad y=14.85, \quad z=23.50$
2.	From $u=1-x$, $\frac{\mathrm{d}u}{\mathrm{d}x}=-1$. Limits: when $x=0, u=1$, and when $x=1, u=0$. Therefore $\int_0^1 x^n(1-x)^m \, \mathrm{d}x=\int_1^0 (1-u)^n u^m (-\mathrm{d}u)$ $=\int_0^1 (1-u)^n u^m \, \mathrm{d}u$ $=\int_0^1 (1-x)^n x^m \, \mathrm{d}x$ (by a change of dummy variables) By substituting $n=2$ and $m=\frac{1}{2}$ into the previous result: $\int_0^1 x^2(1-x)^{\frac{1}{2}} \, \mathrm{d}x=\int_0^1 (1-x)^2 x^{\frac{1}{2}} \, \mathrm{d}x$ $=\int_0^1 (1-2x+x^2)x^{\frac{1}{2}} \, \mathrm{d}x$ $=\int_0^1 x^{\frac{1}{2}}-2x^{\frac{3}{2}}+x^{\frac{5}{2}} \, \mathrm{d}x$ $=\left[\frac{2}{3}x^{\frac{3}{2}}-\frac{4}{5}x^{\frac{5}{2}}+\frac{2}{7}x^{\frac{7}{2}}\right]_0^1=\frac{16}{105}$
3.	Let $\mathrm{P}(z)=z^3-2z^2+az+1+3\mathrm{i}$ $\Rightarrow \mathrm{P}(\mathrm{i})=-1-2-a+1+3\mathrm{i}=0$ $\Rightarrow a=-2+3\mathrm{i}$ Use long division or by comparing coefficient method, $\mathrm{P}(z)=z^3-2z^2+(-2+3\mathrm{i})z+1+3\mathrm{i}$ $=(z-\mathrm{i})\left[z^2+(-2+\mathrm{i})z-3+\mathrm{i}\right]$ $z^2+(-2+\mathrm{i})z-3+\mathrm{i}=0 \Rightarrow z=\frac{-(-2+\mathrm{i})\pm\sqrt{(-2+\mathrm{i})^2-4(-3+\mathrm{i})}}{2}$ $\Rightarrow z=\frac{(2-\mathrm{i})\pm(4-\mathrm{i})}{2} \Rightarrow z=-1 \text{ or } 3-\mathrm{i}$
4.	<u>Method 1 (Algebraic Method)</u>

<table>
<tr><td></td><td>
$\frac{x+6}{x-2} \ge x+1$
$\Rightarrow \frac{x+6-(x+1)(x-2)}{x-2} \ge 0$
$\Rightarrow \frac{-x^2+2x+8}{x-2} \ge 0$
$\Rightarrow \frac{x^2-2x-8}{x-2} \le 0$
$\Rightarrow \frac{(x-4)(x+2)}{x-2} \le 0$


Therefore $x \le -2$ or $2 < x \le 4$

<u>Method 2 (Graphical Method)</u>
$\frac{x+6}{x-2} - (x+1) \ge 0$


$\therefore\ x \le -2$ or $2 < x \le 4$
</td></tr>
<tr><td></td><td>
Replace x with $\frac{1}{x}$.
$\Rightarrow \frac{\frac{1}{x}+6}{\frac{1}{x}-2} \ge \frac{1}{x}+1 \Rightarrow \frac{1+6x}{1-2x} \ge \frac{1+x}{x}$
Using the previous result $\Rightarrow \frac{1}{x} \le -2$ or $2 < \frac{1}{x} \le 4$
$\Rightarrow -\frac{1}{2} \le x < 0$ or $\frac{1}{4} \le x < \frac{1}{2}$
</td></tr>
<tr><td>5.</td><td>
Area $= \frac{1}{2}\pi\left(\frac{a}{2}\right)^2 + ab + \frac{1}{2}a^2 \sin 60^\circ = \frac{\pi a^2}{8} + ab + \frac{\sqrt{3}}{4}a^2$
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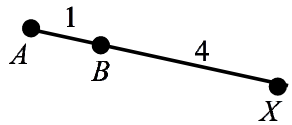
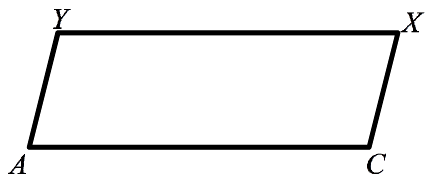
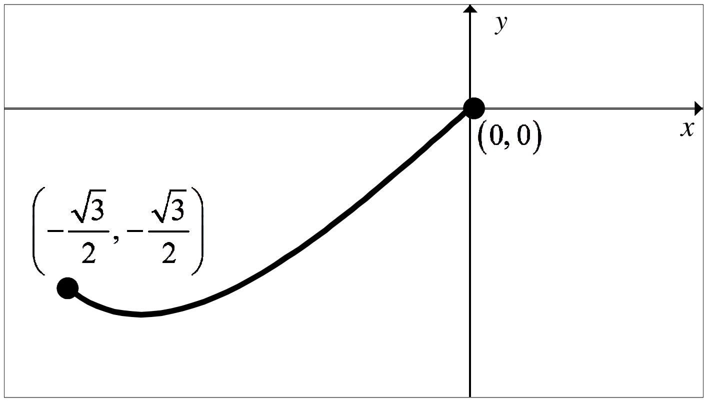
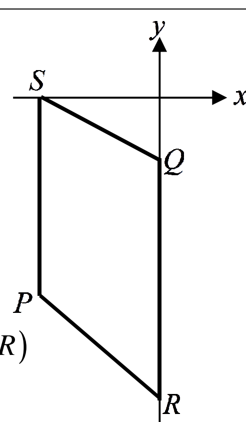
	Thus $\dfrac{\pi a^2}{8}+ab+\dfrac{\sqrt{3}}{4}a^2=400$ $\Rightarrow b=\dfrac{400-\dfrac{\pi a^2}{8}-\dfrac{\sqrt{3}}{4}a^2}{a}=\dfrac{400}{a}-\dfrac{\pi}{8}a-\dfrac{\sqrt{3}}{4}a$ --- (1) Perimeter, $P=2a+2b+\dfrac{\pi a}{2}$ Sub (1) $\quad=2a+2\left(\dfrac{400}{a}-\dfrac{\pi}{8}a-\dfrac{\sqrt{3}}{4}a\right)+\dfrac{\pi a}{2}$ $\quad=\left(2+\dfrac{\pi}{4}-\dfrac{\sqrt{3}}{2}\right)a+\dfrac{800}{a}$ $\dfrac{\mathrm{d}P}{\mathrm{d}a}=2+\dfrac{\pi}{4}-\dfrac{\sqrt{3}}{2}-\dfrac{800}{a^2}$ When $\dfrac{\mathrm{d}P}{\mathrm{d}a}=0$, $\Rightarrow 2+\dfrac{\pi}{4}-\dfrac{\sqrt{3}}{2}-\dfrac{800}{a^2}=0$ $\Rightarrow a^2=\dfrac{800}{2+\dfrac{\pi}{4}-\dfrac{\sqrt{3}}{2}}$ Since $a>0$, therefore $a=\sqrt{\dfrac{800}{2+\dfrac{\pi}{4}-\dfrac{\sqrt{3}}{2}}}$ $=20.416=20.42$ m (correct to 2 d.p) $b=\dfrac{400}{a}-\dfrac{\pi}{8}a-\dfrac{\sqrt{3}}{4}a=2.735=2.74$ m (correct to 2 d.p) Check for minimum using 2nd derivative test: $\dfrac{\mathrm{d}^2P}{\mathrm{d}a^2}=\dfrac{1600}{a^3}>0$ $\therefore$ P is minimum.
6(a)	The decrement follows an AP with first term 60 and common difference -5 Thus area of the island

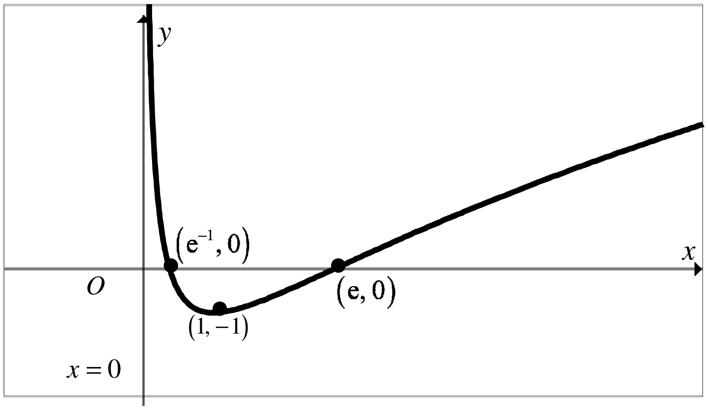
	$=3000-\frac{9}{2}\left[60\times 2+8(-5)\right]=2640\text{ km}^2$
6(b) (i)	The decrement follows a GP with first term 60 and common ratio $\frac{5}{6}$ So total decrement $S_n=\dfrac{60\left[1-\left(\frac{5}{6}\right)^n\right]}{1-\frac{5}{6}}=360-360\left(\frac{5}{6}\right)^n$
6(b) (ii)	$3000-S_n<2720$ $\Rightarrow 3000-\left[360-360\left(\frac{5}{6}\right)^n\right]<2720$ $\Rightarrow 360\left(\frac{5}{6}\right)^n<80$ $\Rightarrow n>\dfrac{\ln\frac{80}{360}}{\ln\frac{5}{6}}=8.25$ Thus at the end of 2023 the area will first fall below the stated amount.
6(b) (iii)	<u>Method 1</u> As $S_\infty=\dfrac{60}{1-\frac{5}{6}}=360$, and r is positive, thus the area decrement will always be less than 360. Hence the area will always be greater than $3000-360=2640\text{ km}^2$ <u>Method 2</u> Area of the island $=3000-S_n=3000-\left[360-360\left(\frac{5}{6}\right)^n\right]$ $=2640+360\left(\frac{5}{6}\right)^n>2640$ for any n. Thus the area will always be greater than 2640 km^2
7(a)	Using cosine rule,

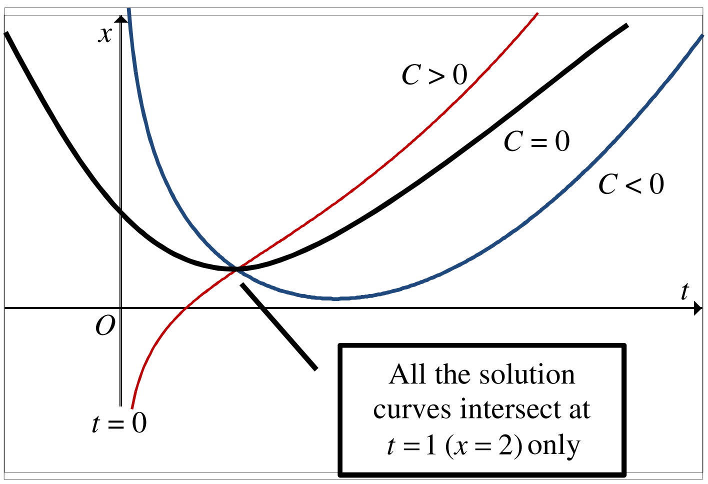
	$AB^2 = 1^2 + \left(\sqrt{2}\right)^2 - 2(1)\left(\sqrt{2}\right)\cos\left(\frac{3\pi}{4} - x\right)$ $= 3 - 2\sqrt{2}\left(\cos\frac{3\pi}{4}\cos x + \sin\frac{3\pi}{4}\sin x\right)$ $= 3 - 2\sqrt{2}\left(-\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)$ $= 3 + 2(\cos x - \sin x)$
	$AB^2 \approx 3 + 2\left(1 - \frac{x^2}{2} - x + \frac{x^3}{3!}\right) = 5 - 2x - x^2 + \frac{1}{3}x^3$ $\boxed{\text{Use } \sin x \approx x - \frac{x^3}{3!} \text{ and } \cos x \approx 1 - \frac{x^2}{2!}}$
7(b)	From $\cos y = hx + kx^2$ Differentiating w.r.t x: $-\sin y \frac{dy}{dx} = h + 2kx$. Differentiating w.r.t x: $-\sin y \frac{d^2y}{dx^2} - \cos y \left(\frac{dy}{dx}\right)^2 = 2k$ <u>Method 2 (Direct Differentiation)</u> From $\cos y = hx + kx^2 \Rightarrow y = \cos^{-1}\left(hx + kx^2\right)$ Differentiating w.r.t x: $\frac{dy}{dx} = \frac{-(h+2kx)}{\sqrt{1-\left(hx+kx^2\right)^2}}$ Differentiating w.r.t x again: $\frac{d^2y}{dx^2} = \frac{-\frac{(h+2kx)^2\left(hx+kx^2\right)}{\sqrt{1-\left(hx+kx^2\right)^2}} - 2k\sqrt{1-\left(hx+kx^2\right)^2}}{1-\left(hx+kx^2\right)^2}$ When $x = 0$, $y = \frac{\pi}{2}$, $\frac{dy}{dx} = -h$ and $\frac{d^2y}{dx^2} = -2k$. Maclaurin series for $y = \frac{\pi}{2} - hx - kx^2 + \ldots$.
7(b)	Replace y with $y+2$, therefore the required series expansion is $y = \frac{\pi}{2} - 2 - hx - kx^2$.

8(i)	Using ratio theorem,  $\overrightarrow{OB}=\dfrac{4\overrightarrow{OA}+\overrightarrow{OX}}{5}$ $\Rightarrow \overrightarrow{OX}=5\overrightarrow{OB}-4\overrightarrow{OA}=5\mathbf{b}-4\mathbf{a}$  Since $ACXY$ forms a parallelogram, we have $\overrightarrow{AC}=\overrightarrow{YX}$ $\Rightarrow \overrightarrow{OY}=\overrightarrow{OX}-\overrightarrow{AC}$ $\Rightarrow \overrightarrow{OY}=(5\mathbf{b}-4\mathbf{a})-\mathbf{b}=4(\mathbf{b}-\mathbf{a})$ Since $\overrightarrow{OY}=4(\mathbf{b}-\mathbf{a})\Rightarrow \overrightarrow{OY}=4\overrightarrow{AB}$ $\Rightarrow \overrightarrow{OY}$ is parallel to $\overrightarrow{AB}\Rightarrow OABY$ is a trapezium.
8(ii)	As $\mathbf{b}$ is a unit vector, thus $\lvert(\mathbf{a}+\mathbf{b})\cdot\mathbf{b}\rvert$ is the length of projection of $\overrightarrow{OC}$ onto $\overrightarrow{OB}$.
8(iii)	Area of $ACXY$ $=\lvert\overrightarrow{AC}\times\overrightarrow{AY}\rvert$ $=\lvert\mathbf{b}\times(4\mathbf{b}-5\mathbf{a})\rvert$ $=5\lvert\mathbf{b}\times\mathbf{a}\rvert$ $=5\times 2=10$ Area of $OAB=\dfrac{1}{2}\lvert\mathbf{b}\times\mathbf{a}\rvert=1$ $\Rightarrow \lvert\mathbf{b}\times\mathbf{a}\rvert=2$ Let the shortest distance be d. Then area of $ACXY$ = $AC\times d$ and $AC=OB=1$. Therefore, $d=10$.

9(i)	
9(ii)	$\frac{\mathrm{d}x}{\mathrm{d}t} = \cos t, \frac{\mathrm{d}y}{\mathrm{d}t} = 2\cos 2t \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{2\cos 2t}{\cos t}$ Coordinates of P are $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ Gradient of tangent at P $= \frac{2}{\sqrt{3}}$ Gradient of normal at P $= -\frac{\sqrt{3}}{2}$ Equation of tangent at P is $y + \frac{\sqrt{3}}{2} = \frac{2}{\sqrt{3}}\left(x + \frac{1}{2}\right)$ $\Rightarrow y = \frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}} - \frac{\sqrt{3}}{2}$ or $y = \frac{2}{\sqrt{3}}x - \frac{1}{2\sqrt{3}}$ Equation of normal at P is $y + \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}\left(x + \frac{1}{2}\right)$ $\Rightarrow y = -\frac{\sqrt{3}}{2}x - \frac{3\sqrt{3}}{4}$
9(iii)	When $x = 0$, $Q = \left(0, \frac{-1}{2\sqrt{3}}\right)$ and $R = \left(0, \frac{-3\sqrt{3}}{4}\right)$. Also, $S = \left(\frac{-1}{2}, 0\right)$.  Area of quadrilateral $PRQS$ = Area of trapezium $= \frac{1}{2}\left(\frac{1}{2}\right)(PS + QR)$ $= \frac{1}{4}\left(\frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{4} - \frac{1}{2\sqrt{3}}\right) = \frac{13}{16\sqrt{3}}$ or 0.469 (3 s.f.)

10(i)	Let $P(n)$ be the statement $u_n = \frac{n}{2^n}, n \geq 1$. When $n=1$, $\text{LHS} = u_1 = \frac{1}{2}$, $\text{RHS} = \frac{1}{2^1}$ $\therefore$ $P(1)$ is true. Assume $P(k)$ is true for some $k \geq 1$, i.e. $u_k = \frac{k}{2^k}$ Prove $P(k+1)$ is true: LHS of $P(k+1) = u_{k+1}$ $= u_k + \frac{1-k}{2^{k+1}}$ Using recurrence relation $= \frac{k}{2^k} + \frac{1-k}{2^{k+1}}$ Using assumption $= \frac{2k+1-k}{2^{k+1}}$ $= \frac{k+1}{2^{k+1}} = \text{RHS of } P(k+1)$ $\therefore P(k+1)$ is true Since $P(1)$ is true and $P(k)$ is true implies $P(k+1)$ is true, by mathematical induction, $P(n)$ is true for $n \geq 1$.
10(ii)	As $n \to \infty, \frac{n}{2^n} \to 0$ Thus the sequence converges to 0.
10(iii)	$\sum_{n=1}^{N} \frac{1-n}{2^{n+1}} = \sum_{n=1}^{N} (u_{n+1} - u_n)$ $= \cancel{u_2} - u_1$ $+\cancel{u_3} - \cancel{u_2}$ $+\cancel{u_4} - \cancel{u_3}$ $+\ldots$ $+u_{N+1} - \cancel{u_N} = u_{N+1} - u_1 = \frac{N+1}{2^{N+1}} - \frac{1}{2}$
10(iv)	Since $\frac{1-n}{2^{n+1}} = \frac{1}{2^{n+1}} - \frac{1}{2}\frac{n}{2^n} = \frac{1}{2^{n+1}} - \frac{1}{2}u_n$ $\Rightarrow \sum_{n=1}^{N} \frac{1-n}{2^{n+1}} = \sum_{n=1}^{N} \left(\frac{1}{2^{n+1}} - \frac{1}{2}u_n \right)$ $\Rightarrow \sum_{n=1}^{N} u_n = 2\left[\sum_{n=1}^{N} \frac{1}{2^{n+1}} - \sum_{n=1}^{N} \frac{1-n}{2^{n+1}} \right]$ --------- (*) $= 2\left[\frac{\frac{1}{2^2}\left(1-\left(\frac{1}{2}\right)^N\right)}{1-\frac{1}{2}} - \frac{N+1}{2^{N+1}} + \frac{1}{2} \right]$ Use of GP sum formula; Use result in (iii) $= 2\left[\frac{1}{2} - \left(\frac{1}{2}\right)^{N+1} - \frac{N+1}{2^{N+1}} + \frac{1}{2} \right] = 2 - \frac{N+2}{2^N}$

11(i)	 To obtain x-intercepts, let $y = 0$ $\Rightarrow (\ln x)^2 = 1$ $\Rightarrow \ln x = \pm 1$ $\Rightarrow x = e^1$ or e^{-1} To obtain the turning point, find $\frac{dy}{dx} = 2\ln x$. Let $\frac{dy}{dx} = 0 \Rightarrow 2\ln x = 0 \Rightarrow x = 1$ Thus coordinates of turning point is $(1, -1)$.
11(ii)	Area of region R $= \int_{e^{-1}}^{e} -\left((\ln x)^2 - 1\right) dx$ $= -\left[x(\ln x)^2\right]_{e^{-1}}^{e} + \int_{e^{-1}}^{e} x \frac{2\ln x}{x} dx + [x]_{e^{-1}}^{e}$ $= -\left(e - e^{-1}\right) + 2\left([x\ln x]_{e^{-1}}^{e} - \int_{e^{-1}}^{e} 1\, dx\right) + \left(e - e^{-1}\right)$ $= -\left(e - e^{-1}\right) + 2\left(\left(e + e^{-1}\right) - \left(e - e^{-1}\right)\right) + \left(e - e^{-1}\right)$ $= 4e^{-1}$
11(iii)	Make x the subject: $y = (\ln x)^2 - 1$ $\Rightarrow \ln x = \pm\sqrt{y+1}$ $\Rightarrow x = e^{\pm\sqrt{y+1}}$ Thus the volume obtained $= \pi \int_{-1}^{0} \left(e^{\sqrt{y+1}}\right)^2 - \left(e^{-\sqrt{y+1}}\right)^2 dy = 12.2$ (to 3 s.f.)
12(a) (i)	$\frac{d}{dt}\left[t\frac{dx}{dt}\right] = t\frac{d^2x}{dt^2} + \frac{dx}{dt}$ ------ (*)

	Note that $t\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\frac{\mathrm{d}x}{\mathrm{d}t}=4t-1 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left[t\frac{\mathrm{d}x}{\mathrm{d}t}\right]=4t-1$ Thus $t\frac{\mathrm{d}x}{\mathrm{d}t}=\int 4t-1\ \mathrm{d}t$ $\Rightarrow t\frac{\mathrm{d}x}{\mathrm{d}t}=2t^2-t+C$ ------ $(**)$ $\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t}=2t-1+\frac{C}{t}$ $\Rightarrow x=t^2-t+C\ln t+D$ where C and D are constants. When $t=1$, $x=2$. $\therefore\ D=2$ $\Rightarrow x=t^2-t+C\ln t+2$
12(a) (ii)	
12(b) (i)	Let $\frac{\mathrm{d}x}{\mathrm{d}t}=ax-bx^2$ where a and b are constants When $x=6, \frac{\mathrm{d}x}{\mathrm{d}t}=0.$ $0=6a-36b \Rightarrow a=6b$ $\therefore \frac{\mathrm{d}x}{\mathrm{d}t}=6bx-bx^2$ $\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t}=kx(6-x)$ where $k=b$.
12(b) (ii)	From $\frac{1}{6x-x^2}\frac{\mathrm{d}x}{\mathrm{d}t}=k$

	<u>Integration Method 1 (Use Completing the square)</u> $\Rightarrow \int \frac{1}{9-(x-3)^2} \; \mathrm{d}x = k\int \mathrm{d}t$ $\Rightarrow \frac{1}{6}\ln\left\lvert\frac{3+(x-3)}{3-(x-3)}\right\rvert = kt + C$ <u>Integration Method 2 (Use Partial Fractions)</u> $\Rightarrow \frac{1}{6}\int \frac{1}{x}+\frac{1}{6-x} \; \mathrm{d}x = k\int \mathrm{d}t$ $\frac{1}{6}\left(\ln\lvert x\rvert - \lvert 6-x\rvert\right) = kt + C$ $\Rightarrow \left\lvert\frac{x}{6-x}\right\rvert = \mathrm{e}^{6kt}\cdot\mathrm{e}^{C}$ $\Rightarrow \frac{x}{6-x} = A\mathrm{e}^{6kt}$ $\Rightarrow x = 6A\mathrm{e}^{6kt} - xA\mathrm{e}^{6kt} \Rightarrow x = \frac{6A\mathrm{e}^{6kt}}{1+A\mathrm{e}^{6kt}}$ When $t=0,\ x=1$, $\therefore A = \frac{1}{5}$ $\Rightarrow x = \frac{6\mathrm{e}^{6kt}}{5+\mathrm{e}^{6kt}}$
12(b) (iii)	Let $G = \frac{\mathrm{d}x}{\mathrm{d}t} = kx(6-x)$ Greatest/Least growth rate occurs when $\frac{\mathrm{dG}}{\mathrm{d}x} = 0$ $\Rightarrow 6k - 2kx = 0 \Rightarrow x = 3$ Since $\frac{\mathrm{d}x}{\mathrm{d}t} = kx(6-x)$ is a quadratic expression where the coefficient of x^2 is negative, the growth rate is the greatest when $x=3$ and $t=6$. $\therefore\ 3 = \frac{6\mathrm{e}^{36k}}{5+\mathrm{e}^{36k}}$ $\Rightarrow \mathrm{e}^{36k} = 5$ $\Rightarrow k = \frac{\ln 5}{36} = 0.04471 \;$ (to 5 d.p)