2014 DHS Year 6 Prelim Examination Paper 1
Suggested Solution

Qn	Suggested Solution
1(i)	m : weight of mackerel in kg s : weight of salmon in kg t : weight of tuna in kg $m + s + t = 800$ $7m + 21s + 39t = 20300$ $5m + 23s + 49t = 23900$ Using GC, $m = 200,\ s = 250,\ t = 350$. Therefore the fisherman has 250 kg of salmon.
(ii)	m : weight of mackerel in kg s : weight of salmon in kg t : weight of tuna in kg $m + s + t = 600$ $7m + 21s + 39t = 20300$ $5m + 23s + 49t = 23900$ Using GC, $m = -460,\ s = 990,\ t = 70$. Since the weight of all fishes must be non-negative, the fisherman's claim is not possible. Or Since the weight of salmon and tuna is more than 600kg, the fisherman's claim is not possible.

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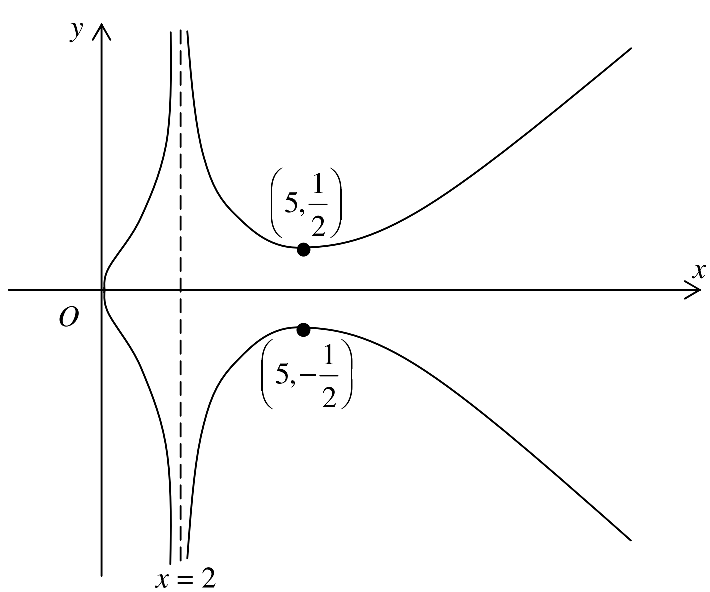
2(a)	

2(b)	$\dfrac{(x-1)^2}{4}+(y-2)^2=1$ Making y the subject of formula: $y=2\pm\sqrt{1-\dfrac{(x-1)^2}{4}}$ Let the volume of solid generated when the curve $y=2+\sqrt{1-\dfrac{(x-1)^2}{4}}$ is rotated about x-axis from $x=2$ to $x=3$ be V_1. $V_1=\int_2^3 \pi y^2\ \mathrm{d}x$ $=\int_2^3 \pi\left(2+\sqrt{1-\dfrac{(x-1)^2}{4}}\right)^2\ \mathrm{d}x$ = 21.593 Let the volume of solid generated when the curve $y=2-\sqrt{1-\dfrac{(x-1)^2}{4}}$ is rotated about x-axis from $x=2$ to $x=3$ be V_2. $V_2=\int_2^3 \pi y^2\ \mathrm{d}x$ $=\int_2^3 \pi\left(2-\sqrt{1-\dfrac{(x-1)^2}{4}}\right)^2\ \mathrm{d}x$ = 6.1573 Volume of required solid $= V_1 - V_2$ = 15.4 (to 3sf)

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3(i)	$\mathrm{e}^y=1+3x+2x^2$ Differentiate with respect to x, $\mathrm{e}^y\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)=3+4x$ Differentiate again with respect to x, $\mathrm{e}^y\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\mathrm{e}^y\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=4$ $\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=4\mathrm{e}^{-y}$ (shown)
3(ii)	$\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=4\mathrm{e}^{-y}$ Differentiating again with respect to x,

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	$2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-4\mathrm{e}^{-y}\frac{\mathrm{d}y}{\mathrm{d}x}$ $\therefore$ when $x=0,\ y=0,\ \frac{\mathrm{d}y}{\mathrm{d}x}=3,\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-5,$ $\mathrm{e}^0(3)^3+3\mathrm{e}^0(3)(-5)+\mathrm{e}^0\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=0$ $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=18$ $\therefore y=0+3x-\frac{5}{2!}x^2+\frac{18}{3!}x^3\cdots=3x-\frac{5}{2}x^2+3x^3\cdots$
3(iii)	$1+3x+2x^2=1.0302$ $x=-1.51$ (reject) or $x=0.01$ $y=\ln\left(1+3x+2x^2\right)=3x-\frac{5}{2}x^2+3x^3+\cdots$ Using $x=0.01$, $\ln\left(1+3(0.01)+2(0.01)^2\right)\approx 3(0.01)-\frac{5}{2}(0.01)^2+3(0.01)^3$ $\ln(1.0302)\approx 0.0298$ (4 d.p.)

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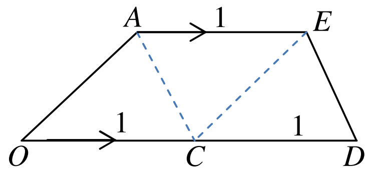
4(i)	$\underset{A}{}\overset{2}{\rule{5em}{0.4pt}}\underset{C}{}\overset{1}{\rule{2.5em}{0.4pt}}\underset{B}{}$ $\overrightarrow{OC}=\frac{\mathbf{a}+2\mathbf{b}}{3}=\frac{\begin{pmatrix}p\\1\\-3\end{pmatrix}+2\begin{pmatrix}1\\0\\0\end{pmatrix}}{3}=\frac{1}{3}\begin{pmatrix}p+2\\1\\-3\end{pmatrix}$ Area of triangle OAC $=\frac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{OC}\right\|$ $=\frac{1}{2}\left\|\begin{pmatrix}p\\1\\-3\end{pmatrix}\times\frac{1}{3}\begin{pmatrix}p+2\\1\\-3\end{pmatrix}\right\|$ $=\frac{1}{6}\left\|\begin{pmatrix}0\\-6-3p+3p\\p-2-p\end{pmatrix}\right\|=\frac{2}{6}\left\|\begin{pmatrix}0\\-3\\-1\end{pmatrix}\right\|$ $=\frac{\sqrt{10}}{3}$

Qn	Suggested Solution
4(ii)	



Triangle *OAD*, triangle *ADE* and triangle *OAC* have the same height and base and thus they have the same area.

Area of trapezium *OAED*

$=3\left(\dfrac{\sqrt{10}}{3}\right)=\sqrt{10}$

4(iii)

$$\cos 135^\circ=\frac{\begin{pmatrix}p\\1\\-3\end{pmatrix}\bullet\begin{pmatrix}1\\0\\0\end{pmatrix}}{\sqrt{p^2+10}\sqrt{1}}$$

$$-\frac{1}{\sqrt{2}}=\frac{p}{\sqrt{p^2+10}}$$

$$p^2+10=2p^2$$

$$p^2=10$$

$$p=-\sqrt{10}\ \ (\text{reject } p=\sqrt{10} \text{ since } \mathbf{a}\bullet\mathbf{b}<0)$$

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5(i)	

$$\sqrt{3}-\mathrm{i}=2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}$$

$$w=2\left(\sqrt{3}-\mathrm{i}\right)z$$

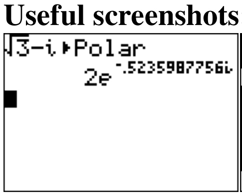
$$=2\left(2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}\right)r\mathrm{e}^{\mathrm{i}\theta}$$

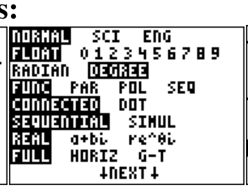
$$=4r\mathrm{e}^{\mathrm{i}\left(\theta-\frac{\pi}{6}\right)}$$

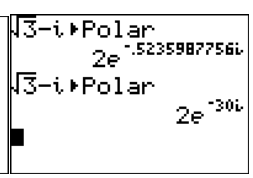
$$|w|=4r$$

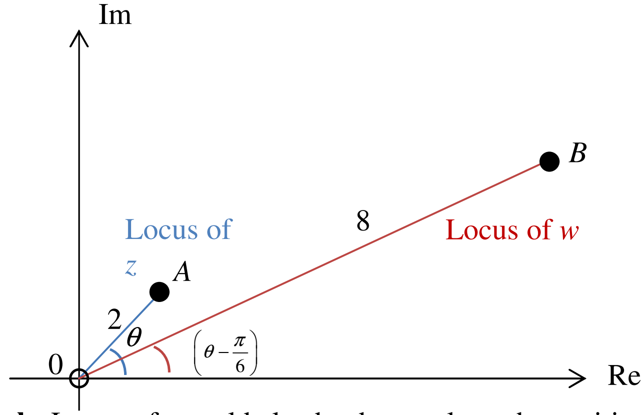
$$\arg w=\theta-\frac{\pi}{6}\ \left(\because \frac{\pi}{6}<\theta\le\frac{\pi}{2}\right)$$

Useful screenshots:







Qn	Suggested Solution
5(ii)	 **Remark:** Locus of z could also be drawn along the positive Im-axis as values of θ include $\frac{\pi}{2}$.
5(iii)	$\left\lvert\frac{w^2}{2z^*}\right\rvert=\frac{\lvert w\rvert^2}{2\lvert z\rvert}=\frac{16r^2}{2r}=8r$ Since $0<r\le 2$, $\therefore 0<\left\lvert\frac{w^2}{2z^*}\right\rvert\le 16.$

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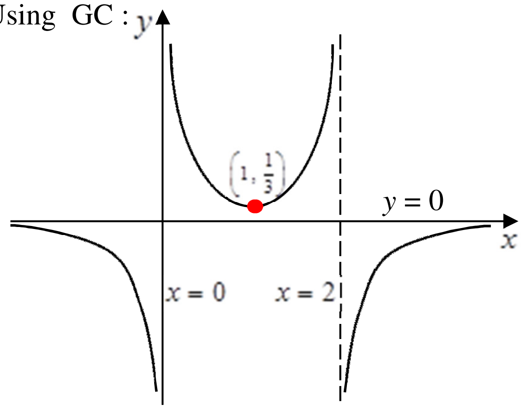
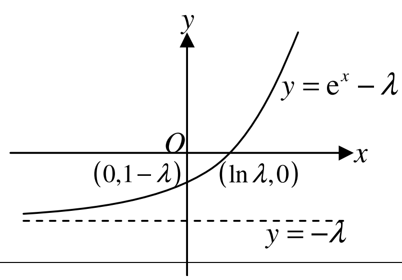
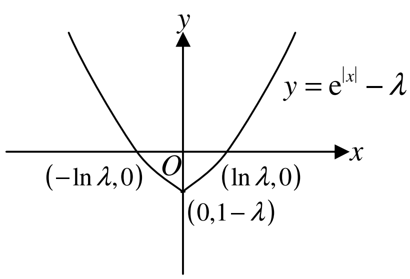
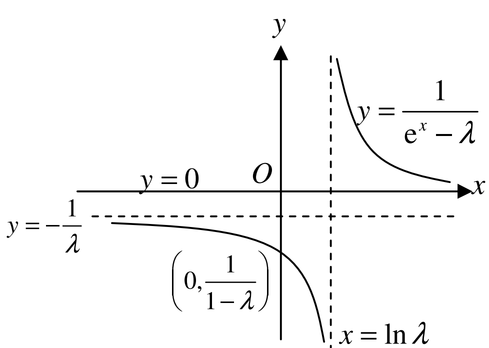
6	$x\frac{\mathrm{d}y}{\mathrm{d}x}+y-3(xy)^2=0$ ....(1) Given $u=xy$: $\frac{\mathrm{d}u}{\mathrm{d}x}=x\frac{\mathrm{d}y}{\mathrm{d}x}+y$ Substitute into (1): $\frac{\mathrm{d}u}{\mathrm{d}x}-3u^2=0$ $\frac{\mathrm{d}u}{\mathrm{d}x}=3u^2$ $\int \frac{1}{u^2}\,du=\int 3\,dx$ $\Rightarrow -\frac{1}{u}=3x+C$ (C arbitrary constant) or $u=-\frac{1}{3x+C}$ $\therefore y=-\frac{1}{x(3x+C)}$ (*)
6(i)	Given $\left(1,\frac{1}{3}\right)$ $\frac{1}{3}=-\frac{1}{1(3+C)}\Rightarrow C=-6$

	Hence, $y=-\dfrac{1}{3x(x-2)}$ Using GC : 
6(ii)	y has no turning point when $C=0$, i.e. particular solution is $y=-\dfrac{1}{3x^2}$

Qn	Suggested Solution
7(i)	 As $x\to-\infty, \mathrm{e}^x\to 0$ $\Rightarrow\ y\to-\lambda$ hence, horizontal asym is $y=-\lambda$
7(ii)	
7(iii)	

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7(iv)	$\left(e^x - e\right)\left(e^{\lvert x\rvert} - e\right) = 1$ $\Rightarrow e^{\lvert x\rvert} - e = \dfrac{1}{e^x - e}$ i.e. $\lambda = e$ Since $\dfrac{1}{1-e} > 1 - e$, from the graphs of $y = e^{\lvert x\rvert} - e$ and $y = \dfrac{1}{e^x - e}$ will intersect 3 times. Thus there will be 3 solutions for $\left(e^x - e\right)\left(e^{\lvert x\rvert} - e\right) = 1$.

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8(a)	$\int x\sec^2(x+a)\,dx$ $= x\tan(x+a) - \int \tan(x+a)\,dx$ $= x\tan(x+a) - \ln\lvert\sec(x+a)\rvert + C$ **OR**: $x\tan(x+a) + \ln\lvert\cos(x+a)\rvert + C$
8(b)	$\int \dfrac{x-1}{x^2-2x+2}\,dx = \dfrac{1}{2}\int \dfrac{2x-2}{x^2-2x+2}\,dx$ $= \dfrac{1}{2}\ln(x^2-2x+2) + C$
8(b) (i)	$\int_1^2 \dfrac{x-4}{x^2-2x+2}\,dx$ $= \int_1^2 \dfrac{x-1}{x^2-2x+2}\,dx - \int_1^2 \dfrac{3}{x^2-2x+2}\,dx$ $= \int_1^2 \dfrac{x-1}{x^2-2x+2}\,dx - \int_1^2 \dfrac{3}{(x-1)^2+1}\,dx$ $= \dfrac{1}{2}\left[\ln\left(x^2-2x+2\right)\right]_1^2 - 3\left[\tan^{-1}(x-1)\right]_1^2$ $= \dfrac{1}{2}\left[\ln 2 - \ln 1\right] - 3\left[\tan^{-1}1 - \tan^{-1}0\right]$ $= \dfrac{1}{2}\ln 2 - \dfrac{3\pi}{4}$
8(b) (ii)	Note that $\dfrac{x-1}{x^2-2x+2} = \dfrac{x-1}{(x-1)^2+1}$: $-$ $+$ (1) $\int_{2-p}^{p} \left\lvert \dfrac{x-1}{x^2-2x+2} \right\rvert dx$ $= -\int_{2-p}^{1} \dfrac{x-1}{(x-1)^2+1}\,dx + \int_1^p \dfrac{x-1}{(x-1)^2+1}\,dx$ $= 2\int_1^p \dfrac{x-1}{(x-1)^2+1}\,dx$ (by symmetry) $=2\left[\dfrac{1}{2}\ln(x^2-2x+2)\right]_1^p = \ln\left(p^2-2p+2\right)$

Qn	Suggested Solution
9(a)	$y_n - y_{n-1}$ $= \log_k x_n + k - (\log_k x_{n-1} + k)$ $= \log_k x_n - \log_k x_{n-1}$ $= \log_k \dfrac{x_n}{x_{n-1}}$ $= \log_k r$ (a constant, where r is the common ratio) Since the difference between any two consecutive terms is a constant, $\{y_n\}$ is an arithmetic sequence.
9b(i)	$1.01(20000 - x) = 20000$ $1.01x = 0.01(20000)$ $x = 198.02$
9b(ii)	(see below)

No. of payments	Amount owed after each payment in the middle of the month
1	$20\,000 - x$
2	$1.01(20\,000 - x) - x$ $= 1.01(20\,000) - 1.01x - x$
3	$1.01[1.01(20\,000) - 1.01x - x] - x$ $= 1.01^2(20\,000) - 1.01^2 x - 1.01x - x$
…	
n	$1.01^{n-1}(20\,000) - x(1.01^{n-1} + 1.01^{n-2} + ... + 1.01 + 1)$ $= 1.01^{n-1}(20\,000) - x\dfrac{1.01^n - 1}{1.01 - 1}$

For the loan to be paid in full after the n^{th} payment,

$$1.01^{n-1}(20\,000) - x\frac{1.01^n - 1}{1.01 - 1} = 0$$

$$1.01^{n-1}(20\,000) = x\frac{1.01^n - 1}{0.01}$$

$$x = \frac{200(1.01^{n-1})}{1.01^n - 1} \quad \text{(shown)}$$

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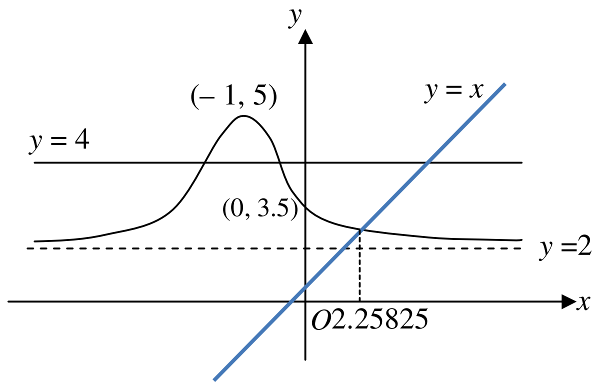
	Alternatively (see table below) For the loan to be paid in full after the n^{th} payment, then $1.01^{n-1}(20\,000) - x\dfrac{1.01(1.01^{n-1}-1)}{1.01-1} - x = 0$ $1.01^{n-1}(20\,000) = x\dfrac{1.01(1.01^{n-1}-1)}{0.01} + x$ $x\dfrac{1.01(1.01^{n-1}-1)+0.01}{0.01} = 1.01^{n-1}(20\,000)$ $x\dfrac{1.01^{n}-1}{0.01} = 1.01^{n-1}(20\,000)$ $x = \dfrac{200(1.01^{n-1})}{1.01^{n}-1}$ (shown)
	For the loan to be fully paid in 3 years (n = 36 months), $x = \dfrac{200(1.01^{36-1})}{1.01^{36}-1}$ $x \approx 657.709$ Hence, for Thomas to fully pay up the loan in exactly 3 years, he should be paying a monthly amount of $657.71

No. of payments	Amount owed at the end of each month
1	$1.01(20\,000 - x)$
2	$1.01[1.01(20\,000 - x) - x]$ $= 1.01^2(20\,000) - 1.01^2 x - 1.01x$
…	
$n-1$	$1.01^{n-1}(20\,000) - x(1.01^{n-1} + 1.01^{n-2} + ... + 1.01)$ $= 1.01^{n-1}(20\,000) - x\left[\dfrac{1.01(1.01^{n-1}-1)}{1.01-1}\right]$

Qn	Suggested Solution
10(i)	$\mathrm{f}: x \mapsto 2+\dfrac{3}{x^2+2x+2},\ x\in\mathbb{R}.$  Since the horizontal line y = 4 passes through the graph of f at two distinct points, f is not one-one, hence f^{-1} does not exist. Note: Horizontal line $y = a$, where $a\in (2,5)$.
10(ii)	For f^{-1} to exist, domain of f is restricted to $x \geq -1$. The smallest value of k is -1.
10(iii)	To find f^{-1} , $y=2+\dfrac{3}{x^2+2x+2}$ $y-2=\dfrac{3}{x^2+2x+2}$ $y-2=\dfrac{3}{(x+1)^2+1}$ $(x+1)^2=\dfrac{3}{y-2}-1$ $x=-1+\sqrt{\dfrac{3}{y-2}-1}$ (reject $x=-1-\sqrt{\dfrac{3}{y-2}-1}\ \because x\geq -1$) Therefore, $\mathrm{f}^{-1}(x)=-1+\sqrt{\dfrac{3}{x-2}-1}$ $\mathrm{D}_{\mathrm{f}^{-1}}=(2,5]$
10(iv)	$\mathrm{D_f} = [-1, \infty)$ $\mathrm{R_f} = (2, 5]$ Since $\mathrm{R_f} \subseteq \mathrm{D_f}$, f^2 exists.

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	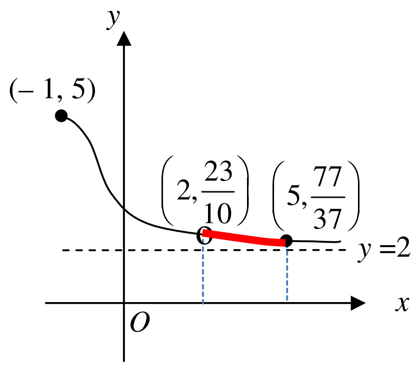 For range of f^2, $[-1,\infty) \to (2,5] \to [\frac{77}{37}, \frac{23}{10})$. When $x = 2$, $y = 2 + \frac{3}{(2)^2 + 2(2) + 2} = \frac{23}{10}$. When $x = 5$, $y = 2 + \frac{3}{(5)^2 + 2(5) + 2} = \frac{77}{37}$.
10(v)	$y = \frac{3}{4x^2 + 4x + 2} \xrightarrow{\text{Replace } x \text{ by } \frac{x}{2}} y = \frac{3}{4\left(\frac{x}{2}\right)^2 + 4\left(\frac{x}{2}\right) + 2}$ $y = \frac{3}{x^2 + 2x + 2}$ $y = \frac{3}{x^2 + 2x + 2} \xrightarrow{\text{Replace } y \text{ by } y\text{-2}} y = 2 + \frac{3}{x^2 + 2x + 2}$ 1. Scale by a factor of 2 parallel to the x-axis. 2. Translate 2 units in the positive y-direction.
10(vi)	$2 + \frac{3}{9x^2 + 6x + 2} < 3x$ $2 + \frac{3}{(3x)^2 + 2(3x) + 2} < 3x$ Replace x by $3x$ to the inequality $2 + \frac{3}{x^2 + 2x + 2} < x$ The range is $3x > 2.25825$ $x > 0.753$ (3 s.f.)

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11(i)	$x = \tan t, \quad y = 2\cos t, \quad \text{for } 0 \le t < \frac{\pi}{2}$ $\frac{\mathrm{d}x}{\mathrm{d}t} = \sec^2 t, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -2\sin t \;\Rightarrow\; \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2\sin t}{\sec^2 t} = -2\sin t\cos^2 t$

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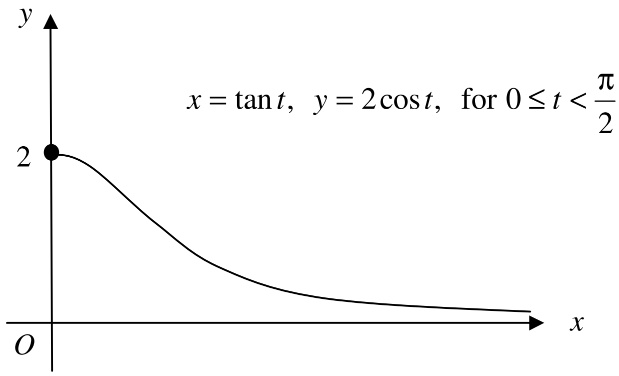
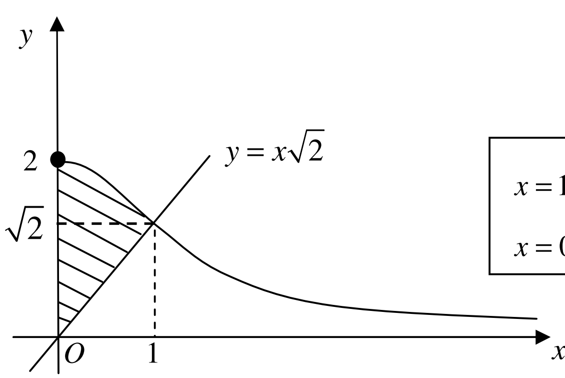
	As $t \to 0,\ \dfrac{\mathrm{d}y}{\mathrm{d}x} \to 0.$ The tangent becomes parallel to the x-axis/tangent is a horizontal line. $x = \tan 0 = 0,\ y = 2\cos 0 = 2$ 
11(ii)	At $P(\tan p,\ 2\cos p)$, gradient of normal $= -\dfrac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}} = -\dfrac{1}{(-2\sin p\cos^2 p)} = \dfrac{1}{2\sin p\cos^2 p},$ **<u>Method 1</u>** Since normal passes through origin, equation of normal : $y = \left(\dfrac{1}{2\sin p\cos^2 p}\right)x$ ..... (1) Since normal intersects curve also at P, substitute $x = \tan p, y = 2\cos p$ into eqn (1) $2\cos p = \dfrac{1}{2\sin p\cos^2 p}(\tan p)$ $= \dfrac{1}{2\cos^3 p}$ $\cos^4 p = \dfrac{1}{4}$ $\cos p = \pm\dfrac{1}{\sqrt{2}}$ $\therefore p = \dfrac{\pi}{4}\ \left(\because 0 < p < \dfrac{\pi}{2}\right)$
	Equation of normal is $y = \dfrac{x}{2\sin\frac{\pi}{4}\cos^2\frac{\pi}{4}}$ $y = \dfrac{x}{2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)^2}$ .... (2) $\therefore y = x\sqrt{2}$ (shown)

Qn	Suggested Solution
	Method 2 Equation of normal : $y-2\cos p=\dfrac{1}{2\sin p\cos^2 p}(x-\tan p)$ ..... (1) Since the normal passes through origin (0,0), substitute $x=0, y=0$ into eqn (1) $0-2\cos p=\dfrac{1}{2\sin p\cos^2 p}(0-\tan p)$ $-4\sin p\cos^3 p=\dfrac{-\sin p}{\cos p}$ $\sin p(4\cos^4 p-1)=0$ $\sin p=0$ or $\cos p=\pm\dfrac{1}{\sqrt{2}}$ $\therefore p=\dfrac{\pi}{4}\left(\because 0<p<\dfrac{\pi}{2}\right)$ Equation of normal which passes through origin is $y-2\cos\dfrac{\pi}{4}=\dfrac{1}{2\sin\dfrac{\pi}{4}\cos^2\dfrac{\pi}{4}}\left(x-\tan\dfrac{\pi}{4}\right)$ $y-2\left(\dfrac{1}{\sqrt{2}}\right)=\dfrac{1}{2\left(\dfrac{1}{\sqrt{2}}\right)\left(\dfrac{1}{\sqrt{2}}\right)^2}(x-1)$ .... (2) $y-\sqrt{2}=\sqrt{2}(x-1)$ $\therefore y=x\sqrt{2}$ (shown)
11(iii)	 $x=1, t=\dfrac{\pi}{4}$ $x=0, t=0$ When $p=\dfrac{\pi}{4}, x=\tan\dfrac{\pi}{4}=1, y=2\cos\dfrac{\pi}{4}=\sqrt{2}$ **Method 1 (with respect to *x*-axis)** Required area $=\int_0^1 y\ \mathrm{d}x-\dfrac{1}{2}bh$ or $\left(\int_0^1 x\sqrt{2}\ \mathrm{d}x\right)$

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	$=\int_0^{\frac{\pi}{4}} 2\cos t\,\left(\sec^2 t\right)\mathrm{d}t-\frac{1}{2}(1)\left(\sqrt{2}\right)$ or $\left[\frac{x^2}{2}\sqrt{2}\right]_0^1$ $=2\int_0^{\frac{\pi}{4}}\sec t\,\mathrm{d}t-\frac{\sqrt{2}}{2}$ $=2\Big[\ln\lvert\sec t+\tan t\rvert\Big]_0^{\frac{\pi}{4}}-\frac{\sqrt{2}}{2}$ $=2\ln\left(\frac{1}{\cos\frac{\pi}{4}}+\tan\frac{\pi}{4}\right)-\frac{\sqrt{2}}{2}$ $=2\ln\left(\sqrt{2}+1\right)-\frac{\sqrt{2}}{2}$ unit^2 **<u>Method 2 (with respect to *y*-axis)</u>** Required area $=\int_{\sqrt{2}}^{2} x\,\mathrm{d}y+\frac{1}{2}bh\quad\left(\int_0^{\sqrt{2}}\frac{y}{\sqrt{2}}\mathrm{d}y\right)$ $\boxed{\begin{gathered}y=2,t=0\\ y=\sqrt{2},t=\frac{\pi}{4}\end{gathered}}$ $=\int_{\frac{\pi}{4}}^{0}\tan t\,\left(-2\sin t\right)\mathrm{d}t+\frac{1}{2}\left(\sqrt{2}\right)(1)$ or $\frac{1}{\sqrt{2}}\left[\frac{y^2}{2}\right]_0^{\sqrt{2}}$ $=2\int_0^{\frac{\pi}{4}}\frac{\sin^2 t}{\cos t}\,\mathrm{d}t+\frac{\sqrt{2}}{2}$ $=2\int_0^{\frac{\pi}{4}}\frac{1-\cos^2 t}{\cos t}\,\mathrm{d}t+\frac{\sqrt{2}}{2}$ $=2\int_0^{\frac{\pi}{4}}\left(\sec t-\cos t\right)\,\mathrm{d}t+\frac{\sqrt{2}}{2}$ $=2\Big[\ln\lvert\sec t+\tan t\rvert-\sin t\Big]_0^{\frac{\pi}{4}}+\frac{\sqrt{2}}{2}$ $=2\ln\left(\frac{1}{\cos\frac{\pi}{4}}+\tan\frac{\pi}{4}-\sin\frac{\pi}{4}\right)+\frac{\sqrt{2}}{2}$ $=2\ln\left(\sqrt{2}+1-\frac{\sqrt{2}}{2}\right)+\frac{\sqrt{2}}{2}$ $=2\ln\left(\sqrt{2}+1\right)-\frac{\sqrt{2}}{2}$ unit^2 Note : Generally $\int\sec t\,\mathrm{d}t=\ln\lvert\sec t+\tan t\rvert$. But in this question where the limits are $0\le t\le\frac{\pi}{4}$, $\int_0^{\frac{\pi}{4}}\sec t\,\mathrm{d}t=\ln\left(\sec t+\tan t\right)$ is acceptable.