CATHOLIC JUNIOR COLLEGE
General Certificate of Education Advanced Level
Higher 2
JC 2 Preliminary Examination

MATHEMATICS 9740/01

Paper 1

28 August 2014

3 hours

Additional Materials:
List of Formulae (MF15)
Graph Paper

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, arrange your answers in NUMERICAL ORDER.
Place this cover sheet in front and fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages, including the cover page.

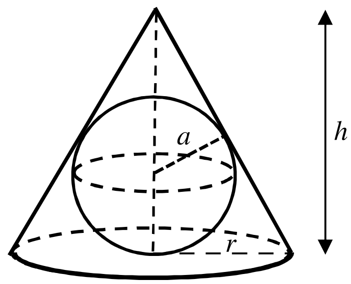
Catholic Junior College

1 Without using a calculator, solve the inequality

$$\frac{x^2(x-1)}{x+2} \geq 0.$$ **[3]**

Hence solve $\dfrac{x^2(|x|-1)}{|x|+2} \geq 0.$ **[2]**

2



The diagram shows a sphere with fixed radius a which is inscribed in a cone with radius r and height h. Show that $r^2 = \dfrac{a^2 h}{h-2a}$ and find the volume of cone, V in terms of a and h.
As h varies, find the value of h that gives the minimum V, leaving your answer in terms of a. (You do not need to verify that V is the minimum.)

[Volume of cone, $V = \dfrac{1}{3}\pi r^2 h$] **[6]**

3 The design of a decorative screen for a Zen garden consists of bamboo poles of decreasing heights, joined together at their sides.

(i) The first pole of the screen has height 2 metres and the heights of the poles form an arithmetic progression such that the 7^{th} pole has height 1.82 metres.
Find the common difference of the arithmetic progression. **[2]**

(ii) It is given that the last pole of the screen has height 1.10 metres.
Show that the total number of poles used for the design of the screen is 31. **[1]**

Unfortunately the manufacturer misunderstands the instructions and constructs the decorative screen wrongly, such that the heights of the poles are in geometric progression with common ratio r.

(iii) Given that the total number of poles used and the heights of the first and last pole remain the same, find r, giving your answer correct to 4 decimal places. **[2]**

(iv) The bamboo poles cost \$20 per metre.
Find the difference in cost between the original and wrongly constructed design. **[3]**

4 Use the method of differences to show that

$$\sum_{n=2}^{N} \frac{1}{n^2-1} = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{N} - \frac{1}{N+1}\right).$$ **[4]**

Find $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ and hence show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{7}{4}$. **[4]**

5 The curve C has parametric equations

$$x = 2 + 2\cos t\ , \quad y = \tan t, \quad \text{for} \quad 0 \le t < \frac{\pi}{2}.$$

(i) Using differentiation, show that the curve C does not have any stationary point. **[2]**

(ii) Sketch C, indicating clearly the equation of the asymptote and coordinates of the x-intercept. You should indicate clearly the feature of the curve near the x-intercept. **[2]**

(iii) A point P on C has parameter $t = \frac{\pi}{4}$.

Find the equation of the tangent at P, leaving your answer in exact form. **[2]**

(iv) Find the cartesian equation of the locus of the mid-point of $(2 + 2\cos t, \tan t)$ and $(-2, 0)$ as t varies. **[3]**

6(a) Prove by mathematical induction that for all positive integers n,

$$\sum_{r=1}^{n} r \ln\left(\frac{r+1}{r}\right) = \ln \frac{(n+1)^n}{n!}.$$ **[5]**

6(b) A sequence of positive real numbers $x_1, x_2, x_3, \ldots$ satisfies the recurrence relation

$$x_{n+1} = \frac{3x_n + 2}{x_n + 1}, \quad \text{for positive integers } n.$$

(i) It is given that as $n \to \infty$, $x_n \to \alpha$. Showing your working clearly, determine the exact value of α. **[2]**

(ii) Use a calculator to determine the behaviour of the sequence when $x_1 = 3.5$. **[1]**

(iii) By considering $x_{n+1} - x_n$, prove that $x_{n+1} < x_n$ if $x_n > \alpha$. **[2]**

7(a) A complex number v is such that $vv^* - 3\sqrt{2}\mathrm{i} + 3\mathrm{i}v = 0$ and $\mathrm{Im}(v) < 2$, where v^* is the conjugate of v.
Find v in the form $x + \mathrm{i}y$, where $x, y \in \mathbb{R}$. **[3]**

7(b) **Do not use a graphic calculator in answering this question.**

(i) Solve the equation $\mathrm{i}w^4 = 3 + \sqrt{3}\mathrm{i}$, giving the roots in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. **[4]**

Given that $-\frac{\pi}{2} \le \arg(w_1) \le 0$ where w_1 is one of the roots.

(ii) Find $\frac{{w_1}^3}{\sqrt{2}}$ in the form $\left(2\sqrt{3}\right)^n (\mathrm{a} + \mathrm{ib})$, where a, b, $n \in \mathbb{R}$. **[3]**

(iii) Find $\arg\left(\frac{\mathrm{i}w_1}{1+\mathrm{i}}\right)$. **[2]**

8 The function f is defined by

$$\mathrm{f} : x \mapsto \frac{x^2 - 2x + 5}{x^2 - 2x - 35}, \quad x \in \mathbb{R}, \quad x \ne 7, \quad x \ne -5.$$

(i) Find $\mathrm{f}'(x)$ and deduce the exact coordinates of any turning points on the graph of $y = \mathrm{f}(x)$. **[3]**

(ii) Sketch the graph of $y = \mathrm{f}(x)$, stating clearly the equations of any asymptotes and the exact coordinates of any turning points and axial intercepts. **[3]**

The function g is defined by

$$\mathrm{g} : x \mapsto ax + b, \quad x \in \mathbb{R},$$

where a and b are positive constants.

(iii) Explain why the composite function gf exists. **[1]**

(iv) Describe a sequence of transformations which transform the graph of $y = \mathrm{f}(x)$ to the graph of $y = \mathrm{gf}(x)$. **[2]**

(v) Show that $(\mathrm{gf})^{-1}$ does not exist and state the largest restriction on the domain of gf in the form $[k, \infty)$ such that $(\mathrm{gf})^{-1}$ exists. **[3]**

[Turn Over

9 The line l_1 passes through the point $(1, -1, 1)$ and is parallel to the vector $2\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$.
The line l_2 has equation $x - 1 = \dfrac{2y+4}{4} = -\dfrac{z}{2}$.

Given that the plane Π_1 contains l_1 and is parallel to l_2 ,

(i) find the Cartesian equation of the plane Π_1, **[4]**

(ii) find the shortest distance between Π_1 and l_2, leaving your answer in exact form. **[3]**

The plane Π_2 has equation $\mathbf{r}.\,(3\mathbf{i} - 2\mathbf{k}) = 1$.

(iii) Find the acute angle between the planes Π_1 and Π_2. **[2]**

(iv) Determine the geometrical relationship between Π_2 and l_1, showing your working clearly. **[2]**

(v) Hence what can be said about the values of a and b such that there is no solution for the following system of linear equations?

$$4x + 7y + 9z = 6,$$
$$3x - 2z = 1,$$
$$2x + y - az = b.$$

[3]

10 Using the substitution $x = a\sin\theta$, where a is a positive constant, show that

$$\int_{\frac{a}{2}}^{a} \sqrt{a^2 - x^2}\ \mathrm{d}x = \frac{a^2}{24}\left(4\pi - 3\sqrt{3}\right).$$

[6]

An ellipse E has the equation $x^2 + 3y^2 = a^2$.

(i) Sketch E, showing clearly the coordinates of any intersections with the axes. **[2]**

(ii) R is the region enclosed by E, the line $y = x$ and the positive x-axis.
Find the exact area of R in the form $k\pi a^2$. **[5]**

(iii) For $x > 0$, S is the region enclosed by E, the line $y = x$ and the positive y-axis.
Using $a = 1$, find the numerical value of the volume of the solid formed when S is rotated through 2π radians about the y-axis. **[3]**

END OF PAPER