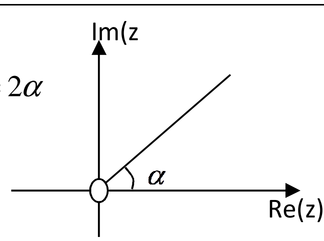
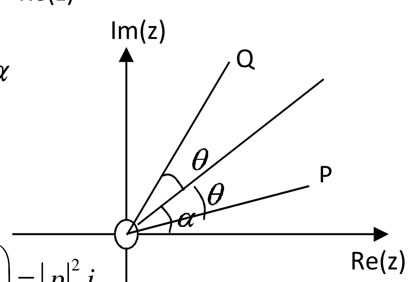
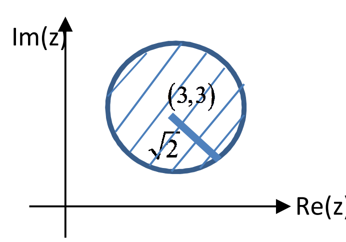
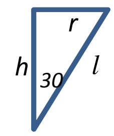
Anderson Junior College
Preliminary Examination 2014
H2 Mathematics Paper 2 (9740/02)

1	Let the number of days the tourist spent in Amaria, Brighten and Costa be x, y, z respectively. 60 x + 40 y + 50 z = 590 30 x + 15 y + 20 z = 280 10 x + 5 y + 15 z = 85 From GC, the solution is x = 8, y = 4 and z = -1. The tourist's calculations were wrong as it is not possible for the number of days to be negative.
2(i)	$\frac{dv}{dt}=\frac{1}{2}(a^2-v^2) \quad \Rightarrow \quad \int\frac{dv}{(a^2-v^2)}=\frac{1}{2}\int dt$ $\Rightarrow \quad \frac{1}{2a}\ln\left\|\frac{a+v}{a-v}\right\|=\frac{1}{2}t+C$ $\Rightarrow \quad \left\|\frac{a+v}{a-v}\right\|=e^{at+2Ca}$ $\Rightarrow \quad \frac{a+v}{a-v}=Ae^{at}$ where $A=\pm e^{2Ca}$ When $t=0$, $v=0$ so $1=A$ $\Rightarrow \quad \frac{a+v}{a-v}=e^{at}$ $\Rightarrow \quad a+v=e^{at}(a-v)$ $\Rightarrow \quad v=\frac{a(e^{at}-1)}{1+e^{at}}$
2(ii)	$v=\frac{a(e^{at}-1)}{1+e^{at}} \Rightarrow v=\frac{a(1-e^{-at})}{e^{-at}+1}$ When $t\to\infty$, $e^{-at}\to 0$, $v\to\frac{a(1-0)}{0+1}=a$ Thus $v_f=a$.
3	$arg\left(5w^2\right)=2\alpha$ $\Rightarrow \arg(5)+2\arg(w)=2\alpha$ $\Rightarrow \arg(w)=\alpha$  (i) $\arg(p)+\arg(q)=(\alpha-\theta)+(\alpha+\theta)=2\alpha$ (ii) $\|p\|=\|q\|$ $\therefore pq=\|p\|\|q\|e^{i2\alpha}=\|p\|^2 e^{i2\alpha}$  When $\alpha=\frac{\pi}{4}$, $\therefore pq==\|p\|^2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=\|p\|^2 i$ Therefore the locus of R is the positive portion of the imaginary axis. $\|z-3-3i\|\le\sqrt{2}$ $\|z-(3+3i)\|\le\sqrt{2}$. The locus is a circle centre at $(3,3)$, radius $\sqrt{2}$.  $\|z-2+ipq\| \quad \Rightarrow \quad \|z-(2-ipq)\|=\left\|z-(2+\|p\|^2)\right\|$ Therefore minimum $\|z-2+ipq\|=3-\sqrt{2}$

4(i)

$$V=\frac{1}{3}\pi r^2 h \Rightarrow V=\frac{1}{3}\pi r^2\left(\sqrt{3}r\right)=\frac{1}{\sqrt{3}}\pi r^3$$



$\frac{r}{h}=\tan 30^o=\frac{1}{\sqrt{3}}$

$$\Rightarrow \frac{\mathrm{d}V}{\mathrm{d}t}=\sqrt{3}\pi r^2\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)$$

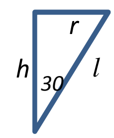
$$\Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t}=\frac{1}{\sqrt{3}\pi r^2}\left(\frac{\mathrm{d}V}{\mathrm{d}t}\right)=\frac{-3}{\sqrt{3}\pi r^2}$$

When $h=5, r=\frac{5}{\sqrt{3}}$

$$\Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t}=\frac{-3}{\sqrt{3}\pi r^2}=\frac{-3\sqrt{3}}{25\pi}$$ cm/s =0.0662 cm/s.

4(ii)

$$S=\pi r l \Rightarrow S=\pi r\left(\frac{r}{\sin 30°}\right)=2\pi r^2$$



$\frac{r}{l}=\sin 30^o=\frac{1}{2}$

$$S=2\pi r^2 \Rightarrow \frac{\mathrm{d}S}{\mathrm{d}r}=4\pi r$$

By chain rule, $\frac{\mathrm{d}S}{\mathrm{d}t}=\frac{\mathrm{d}S}{\mathrm{d}r}\times\frac{\mathrm{d}r}{\mathrm{d}t}$

$$\frac{\mathrm{d}S}{\mathrm{d}t}=4\pi r\times\frac{-3}{\sqrt{3}\pi r^2}=\frac{-4\sqrt{3}}{r}$$

4(iii)

Cylinder : $V=\pi r^2 h=\pi(10)^2(0.81)=81\pi$ cone : $V=\frac{1}{\sqrt{3}}\pi r^3$

$$\Rightarrow 81\pi=\frac{1}{\sqrt{3}}\pi r^3 \qquad \Rightarrow r=\left(81\sqrt{3}\right)^{1/3}=3\sqrt{3}$$

$$\frac{\mathrm{d}S}{\mathrm{d}t}=\frac{-4\sqrt{3}}{3\sqrt{3}}=-\frac{4}{3}$$ cm^2/s

5(i)

(i) $\overrightarrow{OC}=\frac{\boldsymbol{a}+\boldsymbol{b}}{2}$

5(ii)

Let the angle between the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be θ.

$$\boldsymbol{a}\cdot\boldsymbol{b}>0 \Rightarrow |\boldsymbol{a}||\boldsymbol{b}|\cos\theta>0$$

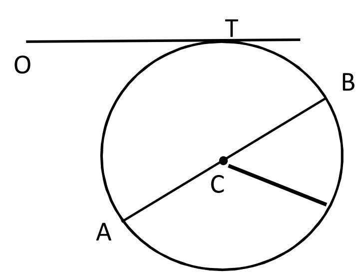
$$\Rightarrow \cos\theta>0$$

$$\Rightarrow 0<\theta<90^o$$

Therefore angle AOB is acute. Since for any point P on or inside the circle with AB as diameter, angle APB $\geq 90^o$, hence O must be outside the circle.

5(iii)

Since T is a tangent to the circle, $\overrightarrow{OT}\perp\overrightarrow{CT}$.

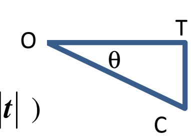


$$\boldsymbol{t}\cdot\left(\boldsymbol{t}-\frac{\boldsymbol{a}+\boldsymbol{b}}{2}\right)=0$$

$$|\boldsymbol{t}|^2-\boldsymbol{t}\cdot\left(\frac{\boldsymbol{a}+\boldsymbol{b}}{2}\right)=0$$

$$\boldsymbol{t}\cdot\left(\frac{\boldsymbol{a}+\boldsymbol{b}}{2}\right)=|\boldsymbol{t}|^2$$

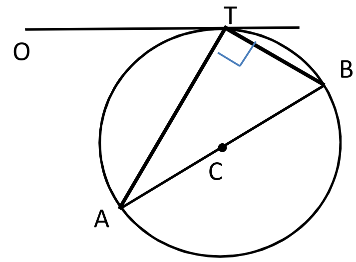
Alternative :



$\boldsymbol{t}\cdot\boldsymbol{c}=|\boldsymbol{t}||\boldsymbol{c}|\cos\theta=|\boldsymbol{t}|^2$ (as $|\boldsymbol{c}|\cos\theta=|\boldsymbol{t}|$)

5(iv)

Since T is on the circle, $\overrightarrow{AT} \perp \overrightarrow{BT}$.



$$(\boldsymbol{t}-\boldsymbol{a})\bullet(\boldsymbol{t}-\boldsymbol{b})=0$$

$$|\boldsymbol{t}|^2-\boldsymbol{t}\bullet\boldsymbol{b}-\boldsymbol{t}\bullet\boldsymbol{a}+\boldsymbol{a}\bullet\boldsymbol{b}=0$$

$$|\boldsymbol{t}|^2 - 2|\boldsymbol{t}|^2 + \boldsymbol{a}\bullet\boldsymbol{b}=0$$

$$|\boldsymbol{t}|^2 = \boldsymbol{a}\bullet\boldsymbol{b} \quad \Rightarrow \quad |\boldsymbol{t}| = (\boldsymbol{a}.\boldsymbol{b})^{\frac{1}{2}}$$

Area of triangle OTC

$$= \frac{1}{2}\left|\boldsymbol{t}\times\left(\frac{\boldsymbol{a}+\boldsymbol{b}}{2}\right)\right|$$

$$= \frac{1}{2}(\boldsymbol{a}.\boldsymbol{b})^{\frac{1}{2}}\left|\frac{\boldsymbol{b}-\boldsymbol{a}}{2}\right|$$

Therefore $|\boldsymbol{t}\times(\boldsymbol{a}+\boldsymbol{b})| = (\boldsymbol{a}.\boldsymbol{b})^{\frac{1}{2}}\,|\boldsymbol{b}-\boldsymbol{a}|$

6(i)

Quota sampling .

No, it does not give a random sample of 50 students as the class sizes range from 25 to 35, so students from smaller classes have a higher chance of being selected than those from bigger classes.

6(ii)

Systematic sampling :

Firstly, find the sampling interval = $\dfrac{300}{50}=6$

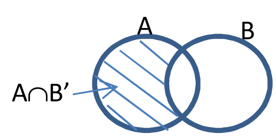
Next, select the random start from the first interval of 6 students, eg 5th student is chosen. Then select the 5th, 11th,17th, 23rd,.. 299th students from the list of 300 students to obtain a random sample of 50 students.

7(i)

$$P(A \mid B') = \frac{1}{2}$$

$$\Rightarrow \frac{P(A\cap B')}{P(B')}=\frac{1}{2}$$

$$\Rightarrow P(A\cap B') = \frac{1}{2}\left(1-P(B)\right)=\frac{1}{2}\left(\frac{1}{4}\right)=\frac{1}{8}$$



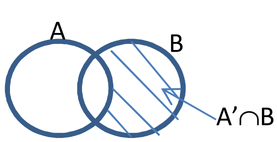
$$P(A\cap B) = P(A) - P(A\cap B')$$

$$= \frac{3}{5} - \frac{1}{8}$$

$$= \frac{19}{40} \text{ (ans)}$$

7(ii)

(ii) $P(A' \mid A\cup B) = \dfrac{P(A'\cap B)}{P(A\cup B)} = \dfrac{P(B)-P(A\cap B)}{P(A)+P(B)-P(A\cap B)}$



$$= \frac{\frac{3}{4}-\frac{19}{40}}{\frac{3}{5}+\frac{3}{4}-\frac{19}{40}}$$

$$= \frac{11}{35}$$

7(iii)

(iii) A independent of C

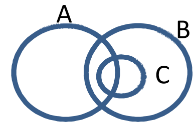
$\Rightarrow P(A'\cap C) = P(A') \,.\, P(C)$

$$= \frac{2}{5}\times\frac{1}{2} \qquad = \frac{1}{5}$$

7(iv)

P($A' \cap B \cap C$) is greatest when $A' \cap C$ is totally in B.

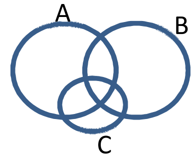
P($A' \cap C$) $= \frac{1}{5} \Rightarrow$ P($A' \cap B \cap C$) $\le \frac{1}{5}$



P($A' \cap B \cap C$) is least when $A' \cap C$ is furthest away from B.

Since P($A' \cap B'$) = 1/8

P($A' \cap B \cap C$) $\ge \frac{1}{5} - \frac{1}{8} = \frac{3}{40}$



$\therefore \quad \frac{3}{40} \le$ P($A' \cap B \cap C$) $\le \frac{1}{5}$

<u>Alternative method:</u>

$P(A \cup B \cup C) \le 1$

$P(A \cup B) + P(A' \cap C) - P(A' \cap B \cap C) \le 1$

$P(A' \cap B \cap C) \ge \frac{7}{8} + \frac{1}{5} - 1$

$P(A' \cap B \cap C) \ge \frac{3}{40}$

8(i)

Family – 3 , Holiday – 4 , Friends – 7

Choose 6 images - at least 3 images from the 'Friends' folder and at least 1 image from each of the other two folders.

<u>Case 1 : 4 Friends +1 Holiday +1 Family</u> .

No of selections = ${}^7C_4 \times {}^4C_1 \times {}^3C_1 = 420$

<u>Case 2 : 3 Friends + 2 Holiday + 1 Family .</u>

No of selections = ${}^7C_3 \times {}^4C_2 \times {}^3C_1 = 630$

<u>Case 3 : 3Friends +1 Holiday + 2 Family</u>

No of selections = ${}^7C_3 \times {}^4C_1 \times {}^3C_2 = 420$

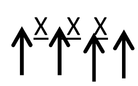
Total no of selections = 420 +420 +630 = <u>1470</u>

8(ii)

<u>Case 1</u> : identical images in first row : <u>P</u> <u>X</u> <u>P</u>

No of arrangements = 6 x 5! Or 6! = 720

<u>Case 2 :</u> identical images in second row of 5



No of arrangements = ${}^4C_2 \times 6! = 4320$

Total no of arrangements = 720 + 4320 = <u>5040</u>

8(iii)

P(image A will appear a second time in the next n images) > 0.9

$$\Rightarrow \frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} + \left(\frac{2}{3}\right)^{n-1}\cdot\frac{1}{3} > 0.9$$

$$\frac{\frac{1}{3}\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\frac{2}{3}} > 0.9$$

$$\Rightarrow \left(\frac{2}{3}\right)^n < 0.1$$

$$\Rightarrow \quad \Rightarrow n\ln\left(\frac{2}{3}\right) < \ln 0.1$$

$$\Rightarrow n > 5.6789$$

$$\Rightarrow least\ n = 6$$

<u>Alternative Method:</u>

P(image A will appear a second time in the next n images) > 0.9

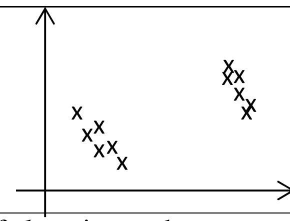
$\Rightarrow$ 1 – P(image A does not appear in the next n images) >0.9

$$\Rightarrow 1-\left(\frac{2}{3}\right)^n > 0.9$$

$$\Rightarrow \left(\frac{2}{3}\right)^n < 0.1$$

9(i)	unbiased estimate of population variance , $s^2 = \frac{10}{9}(10)^2 = \frac{1000}{9}$, and sample mean = $\bar{x}$ Given $H_0 : \mu = 250$ vs $H_1 : \mu \neq 250$, use a t-test at 5% level , Since H_0 is rejected , $P\left(\overline{X} < \bar{x}\right) < 0.025 \quad or \quad P\left(\overline{X} > \bar{x}\right) < 0.025$ $P\left(\overline{X} < \bar{x}\right) < 0.025 \quad or \quad P\left(\overline{X} < \bar{x}\right) > 0.975$ $\Rightarrow \quad P\left(T < \frac{\bar{x}-250}{\sqrt{\frac{1000/9}{10}}}\right) < 0.025 \quad or \quad P\left(T < \frac{\bar{x}-250}{\sqrt{\frac{1000/9}{10}}}\right) > 0.0975$ $\Rightarrow \bar{x} - 250 < -2.2622\sqrt{\frac{1000/9}{10}} \quad or \quad \bar{x} - 250 > 2.2622\sqrt{\frac{1000/9}{10}}$ $\Rightarrow \bar{x} < 250 - 2.2622\sqrt{\frac{1000/9}{10}} \quad or \quad \bar{x} > 250 + 2.2622\sqrt{\frac{1000/9}{10}}$. $\Rightarrow \bar{x} < 242.50 \qquad or \qquad \bar{x} > 257.54$
9(ii)	'5% level of significance' is the probability of 0.05 of concluding that the mean height of the bounce of a tennis ball is not 2.5m when it actually is.
9(iii)	$\sum(x-240) = 545 \,; \quad \sum(x-240)^2 = 5456$ $\bar{x} = \frac{545}{60} + 240 = 249.083 \;; \quad s^2 = \frac{1}{59}\left(5456 - \frac{545^2}{60}\right) = 8.56921$ To test : $H_0 : \mu = 250$ vs $H_1 : \mu < 250$ Use a one-tailed Z-test at 4% level, ie reject H_0 if $p < 0.04$. Under H_0, test statistic , $Z = \frac{\overline{X} - 250}{s/\sqrt{n}} \sim N(0,1)$ From G.C, p-value = 0.00764 < 0.04 , we reject H_0 and conclude at 4% level there is sufficient evidence that the manufacturer's claim is not justified.

10(i)	$P(X > 150) = P(X < 100) = 0.15$. μ = mid point of 100 & 150 = 125 $P(X < 100) = 0.15 \;\Rightarrow\; P\left(Z < \frac{100-125}{\sigma}\right) = 0.15$ $\Rightarrow \frac{-25}{\sigma} = -1.0364338$ $\Rightarrow \quad \sigma = 24.1212 = 24.1$, correct to 3 significant figures.
10 (ii)	Profit, $W = \frac{2}{100}\left(X_1 + X_2 + \ldots\ldots + X_{250}\right) + \frac{1}{100}\left(Y_1 + Y_2 + \ldots\ldots + Y_{400}\right) - 300$
10 (iii)	$E(W) = \frac{1}{50}(250 \times 125) + \frac{1}{100}(400 \times 93.3) - 300 = 698.20$ $Var(W) = \frac{1}{50^2}\left(250 \times 24.1^2\right) + \left(\frac{1}{100}\right)^2\left(400 \times 12.8^2\right) = 64.6346$ 64.73683 if use $\sigma = 24.1212$

	W~N(698.2, 64.6346) P(W > 700) = 0.41142 = 0.411 to 3sf 0.411489 if use 64.73683
10 (iv)	Let S be the number of days out of 60 days where profit exceeds \$700 S ~B(60,0.411) Since 60 is large and np = 24.66 >5, nq > 5, by normal approximation, S~N(24.66, 14.52474) P(S < n) ≥ 0.9 P(S < n-0.5) ≥ 0.9 (cc) stanardise $P\left(Z<\frac{n-0.5-24.66}{\sqrt{14.52474}}\right)\geq 0.9$ $\frac{n-25.16}{\sqrt{14.52474}}\geq 1.28155$ $\Rightarrow \quad n\geq 30.044$. Least n = 31 Alternatively, P(S < n-0.5) ≥ 0.9 By using GC, n – 0.5 ≥ 29.54 $n \geq 30.044$
11 (a)	As seen in the diagram, r value is positive for the 12 bivariate data. Hence r need not be negative. 
11 (b)	The statement is incorrect as insufficient amount of sleep is not the cause of weight gain even though they are negatively linearly correlated. [correlation does not imply causation]
11 (c)	(i) As seen from the data, the increase in mass per day is not a constant. (ii) $x = Ae^{Bt}$ $\Rightarrow$ ln x = lnA + Bt By GC, ln x = 0.25304t+2.665865 ln A = 2.665865 $\Rightarrow$ A = 14.38, B = 0.2530 (to 4 sig fig) (iii) The value of A is the mass of the animal at birth. (iv) If x = 100, ln100 = 0.25304t+2.665865 $\Rightarrow$ t = 7.66 days This model is not suitable for long term prediction as the mass of the amimal cannot increase indefinitely with time.
12 (i)	Let X be number of granules in a bottle. $X \sim P_o(1.3)$ Let Y be number of air bubbles in a bottle. $Y \sim P_o(0.1)$ $X+Y \sim P_o(1.4)$ p = P(a bottle is rejected) = P(X+Y ≥ 3) = 1 – P(X+Y ≤ 2) = 1 – 0.8335 = 0.1665 ≈ 1/6
12 (ii)	P(a rejected bottle contains no air bubble) = P(Y = 0 | X+Y ≥ 3) $= \frac{P(Y=0)P(X\geq 3)}{0.1665}$ $= \frac{0.90484(1-P(X\leq 2))}{0.1665}$ $= \frac{0.90484(1-0.85711)}{0.1665}$

0.776 if use σ = 1/6

	=0.777 (to 3sf)
12 (iii)	$P\left(\left\|\overline{X}-\overline{Y}\right\|\geq 1\right) = 1-P\left(-1<\overline{X}-\overline{Y}<1\right)$ $\overline{X} \sim N(1.3,\frac{1.3}{80})$ and $\overline{Y} \sim N(0.1,\frac{0.1}{80})$ by central limit theorem $\overline{X}-\overline{Y} \sim N(1.2,\frac{1.3}{80}+\frac{0.1}{80})$ $P\left(\left\|\overline{X}-\overline{Y}\right\|\geq 1\right) = 1-P\left(-1<\overline{X}-\overline{Y}<1\right)$ $= 1 - 0.065285 = 0.935$
12 (iv)	Let W be the number of bottles being rejected in a month of 80. $W \sim Bin(80,\frac{1}{6})$ $P_r = {}^{80}C_r\left(\frac{1}{6}\right)^r\left(\frac{5}{6}\right)^{80-r}$ $\frac{P_{r+1}}{P_r} = \frac{{}^{80}C_{r+1}\left(\frac{1}{6}\right)^{r+1}\left(\frac{5}{6}\right)^{80-(r+1)}}{{}^{80}C_r\left(\frac{1}{6}\right)^r\left(\frac{5}{6}\right)^{80-r}}$ $= \frac{\frac{80!}{(r+1)!(80-r-1)!}\left(\frac{1}{6}\right)}{\frac{80!}{r!(80-r)!}\left(\frac{5}{6}\right)} \qquad = \frac{80-r}{5(r+1)}$ If $P_{r+1} > P_r \quad \Rightarrow \frac{80-r}{5(1+r)} > 1 \quad \Rightarrow 80-r > 5r+5 \quad \Rightarrow r < 12.5$ When r ≤ 12, $P_{r+1} > P_r$. so $P_{13} > P_{12} > P_{11} > \ldots\ldots > P_1 > P_0$ When r ≥ 13, $P_{r+1} < P_r$. so $P_{80} < \ldots\ldots < P_{14} < P_{13}$ Hence the most likely number of bottles being rejected is 13.