Section A: Pure Mathematics (40 marks)

1 A tourist in Rome stayed in 3 different hotels throughout his trip. After returning from his trip, he tabulated the amount that he spent per day on hotel tariffs, meals and miscellaneous expenses in each hotel which is as shown below.

Hotel	Hotel Tariffs ($)	Meal Cost ($)	Miscellaneous Expenses ($)
Amaria	60	30	10
Brighten	40	15	5
Costa	50	20	15

The tourist's records of the trip indicated a total of $590 spent for hotel tariffs, $280 for meal expenses and $85 for miscellaneous expenses. By considering the number of days the tourist spent in each hotel, do you think the tourist's calculations were correct? Why? [4]

2 A particle which falls through a thick fluid has a velocity of v at time t. The rate of increase in the velocity at time t is given by $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{1}{2}\left(a^2 - v^2\right)$, where a is a positive constant.

If the velocity of the particle is initially zero and if v_f is the value of v when $t \to \infty$,

(i) Show that $v = \dfrac{a\left(e^{at} - 1\right)}{1 + e^{at}}$ [5]

(ii) Find v_f in terms of a. [2]

3 Sketch the locus of the point W in the Argand diagram representing the complex number w, where $\arg\left(5w^2\right) = 2\alpha$, α is a constant and $0 < \alpha < \dfrac{\pi}{2}$. [2]

The point P represents the complex number p where $0 < \arg(p) < \alpha$. The point Q, which represents the complex number q, is the reflection of point P about the locus of W.

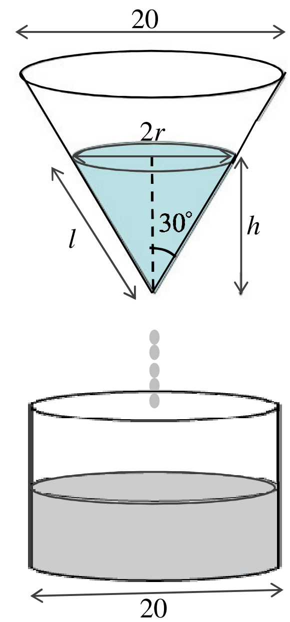
(i) Show that $\arg(p) + \arg(q) = 2\alpha$. [1]

(ii) Deduce that $pq = |p|^2 e^{i2\alpha}$ [1]

Let $\alpha = \dfrac{\pi}{4}$ and point R represents the complex number pq. Describe the locus of R as p varies. [2]

On a separate Argand diagram, sketch the locus given by $|z - 3 - 3i| \le \sqrt{2}$ and find the minimum value of $|z - 2 + ipq|$ [4]

4 An experiment is conducted using the conical filter which is held with its axis vertical as shown. The filter has a radius of 10cm and semi-vertical angle 30°. Chemical solution flows from the filter into the cylindrical container, with radius 10cm, at a constant rate of $3\text{ cm}^3/\text{s}$. At time t seconds, the amount of solution in the filter has height h cm and radius r cm.



(i) Find the rate of decrease of radius of the solution in the filter when $h = 5\text{cm}$. [4]

(ii) Let S denotes the curved surface area of filter in contact with the solution.
Show that $\dfrac{\mathrm{d}S}{\mathrm{d}t} = -\dfrac{4\sqrt{3}}{r}$ cm^2/s. [3]

(iii) When the height of solution in the cylindrical container measures 0.81 cm, volumes of solution left in the filter and the container are the same. Find the rate of change of S at this instant. [3]

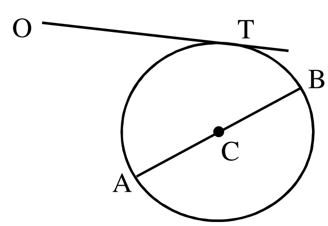
[The volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$ and the curved surface area is πrl where l is the slant height of the cone.]

5 Relative to an origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}\bullet\mathbf{b} > 0$. A and B also lie on a circle with centre C and AB as diameter.

(i) Write down the position vector of C in terms of $\mathbf{a}$ and $\mathbf{b}$. [1]

(ii) Show that the origin O is outside the circle. [2]

T is a point on the circle with position vector $\mathbf{t}$ and OT is a tangent to the circle.



(iii) Show that $\mathbf{t}\bullet\left(\dfrac{\mathbf{a}+\mathbf{b}}{2}\right) = |\mathbf{t}|^2$; [2]

(iv) By considering the triangle ATB, show that the length of OT is given by $(\mathbf{a}\bullet\mathbf{b})^{\frac{1}{2}}$. [2]

By considering the area of triangle OTC, show that $|\mathbf{t}\times(\mathbf{a}+\mathbf{b})| = (\mathbf{a}.\mathbf{b})^{\frac{1}{2}}\,|\mathbf{b}-\mathbf{a}|$. [2]

Section B: Statistics (60 marks)

6 The Students' Council of Abraham College would like to seek the opinion of 50 students of the 10 graduating classes of class sizes ranging from 25 to 35 students each, on the venue and programme of the graduation ball to be held at the end of the year. The total population of the 10 graduating classes is 300 students.

(i) 5 students from each of the 10 classes were randomly selected to be surveyed. State the sampling method used and explain if this method gives a random sample of 50 students from the graduating classes. [2]

(ii) The Council President suggested using systematic sampling instead. Describe how a sample of 50 students can be selected using this method. [1]

7 For the events A and B , $\mathrm{P}(A) = \dfrac{3}{5}$, $\mathrm{P}(B) = \dfrac{3}{4}$ and $\mathrm{P}(A \mid B') = \dfrac{1}{2}$.

Find

(i) $\mathrm{P}(A \cap B)$. [2]

(ii) $\mathrm{P}(A' \mid A \cup B)$ [3]

For a third event C, $\mathrm{P}(C) = \dfrac{1}{2}$ and C is independent of A.

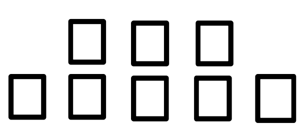
(iii) Find $\mathrm{P}(A' \cap C)$. [1]

(iv) Hence, show that $\dfrac{3}{40} \le P(A' \cap B \cap C) \le \dfrac{1}{5}$. [2]

8 Mary saves her digital images on her computer in three separate folders named 'Family', 'Holiday' and 'Friends'. Her 'Family' folder contains 3 images, her 'Holiday' folder contains 4 images and her 'Friends' folder contains 7 images. All the images are different.

(i) Find the number of different ways in which Mary can choose 6 of these images if there are at least 3 images from the 'Friends' folder and at least 1 image from each of the other two folders. [3]

(ii) Mary decides to choose all the 7 images from her 'Friends' folder and copied 1 particular image once, making it a total of 8 images. She then arranges the 8 images in 2 rows as shown below.



Find the number of arrangements of the images if the 2 identical images must be arranged in the same row separated from each another. [3]

(iii) Mary chooses three images, namely *A, B and C* to form the screensaver on her computer screen. The screensaver refreshes itself every 30seconds, either to a new image or the existing one. Currently, image A is being displayed. Find the least value of n such that the probability that image A appears a second time in the next n images is more than 0.9. [3]

9 A factory that manufactures tennis balls claims that the balls it produced are able to bounce 2.5m high when dropped from a standard height onto a hard surface. A distributer decides to test the claim by randomly selecting a sample of 10 tennis balls from a batch and testing the height of a bounce, X cm of each tennis ball from the sample. It has been found that the unbiased estimate of the variance of X is $\frac{1000}{9}$ cm. The null hypothesis of H_0: $\mu = 250$ is tested at 5% level of significance, against the alternative hypothesis H_1 : $\mu \neq 250$ where μ cm represents the population mean.

(i) Find the range of values of $\bar{x}$, where $\bar{x}$ cm represents the sample mean, if the hypothesis test has concluded that $\mu \neq 250$ at the 5% level of significance. Give your answers correct to 2 decimal places. [3]

(ii) What is meant by '5% level of significance' in the context of this question? [1]

Subsequently, the manufacturer made improvements to the manufacturing process and claimed that the bounce of the tennis balls produced now is at least 2.5m high. The distributer was not convinced and decided to test another random sample of 60 tennis balls and the results were summarised as follows:

$$\sum(x-240)=545\,; \quad \sum(x-240)^2=5456\,.$$

(iii) Test the manufacturer's claim at 4% significance level. [5]

10 A butcher shop prepares 250 beef sausages and 400 chicken sausages per day. The masses of beef sausages and chicken sausages may be assumed to follow independent normal distributions, with mean and standard deviations as shown in the following table.

	Mean (grams)	Standard deviation (grams)
Mass of one beef sausage, X	μ	σ
Mass of one chicken sausage, Y	93.3	12.8

(i) Given that $P(X > 150) = P(X < 100) = 0.15$, state the value of μ and show that $\sigma = 24.1$, correct to 3 significant figures. [3]

(ii) The sausages are sold by weight. Beef and chicken sausages are sold at a profit of \$2 and \$1 per 100 grams respectively. A fixed cost of \$300 per day is incurred regardless of the sales per day. Write down an expression of the daily profit, W, assuming that all sausages are sold every day. [1]

(iii) Show that the probability that the daily profit exceeds \$700 is 0.411. [3]

(iv) The owner of the shop decides to monitor the daily profit for 60 days. He will raise the prices of the sausages if there are less than n days where daily profit exceeds \$700. Find, using suitable approximation, the least value of n if there is a probability of at least 0.9 that he will raise the prices. [4]

11 (a) The linear product correlation coefficient between 2 variables X and Y is denoted by r. A set of 6 bivariate data yields $r=-0.9$ and a second set of 6 different bivariate data also yields $r=-0.9$. Explain, with the aid of a diagram, whether this implies that r is also negative for the combined set of 12 bivariate data. [2]

(b) It is observed in a one-year study that the linear correlation coefficient between the weight gain and the number of sleeping hours per day is close to -1. Comment briefly upon this statement: *"Since the linear correlation coefficient is close to -1, we can therefore conclude that the weight gain is caused by insufficient amount of sleep per day."* [1]

(c) The following table shows the mass of a small animal from the time of birth

No. of days from birth, t	1	2	3	4	5	6
Mass in grams, x	21.6	22.5	28.0	35.2	49.8	75.3

(i) Without using a scatter diagram, give a reason why t and x are not linearly related. [1]

(ii) It is suggested that a suitable model is of the form $x = Ae^{Bt}$, find, correct to 4 significant figures, the estimate values of A and B. [2]

(iii) Give an interpretation, in context, of the value of A. [1]

(iv) Use the suggested model to predict the number of days from birth if the mass is found to be 100 grams. Comment on the suitability of this model for long term prediction. [2]

12 The number of granules and the number of air bubbles in a bottle made by a glass-blower have independent Poisson distributions with mean 1.3 and 0.1 respectively. A bottle will be rejected if the total number of granules and air bubbles is at least 3.

(i) Show that the probability, p, that a bottle has to be rejected is approximately $\frac{1}{6}$. [2]

(ii) Find the probability that a rejected bottle contains no air bubble. [2]

The glass-blower makes 80 bottles in a month.

(iii) Find the probability that the difference between the average number of granules and the average number of air bubbles per bottle in a month is at least 1. [3]

(iv) Denoting the probability that exactly r bottles have to be rejected in a month as P_r and taking the value of p in (i) to be exactly $\frac{1}{6}$, write down an expression for P_r and show that $\frac{P_{r+1}}{P_r} = \frac{80-r}{5(1+r)}$. [2]

Hence, deduce the most likely number of bottles being rejected in a month, giving your reasoning clearly. [2]

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