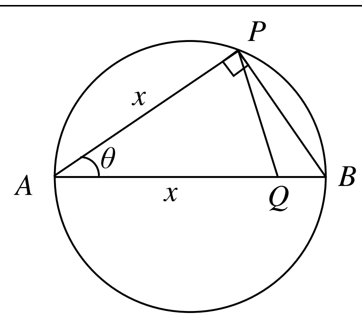
Anglo-Chinese Junior College

H2 Mathematics 9740

2014 JC 2 Preliminary Exam P1 Solution

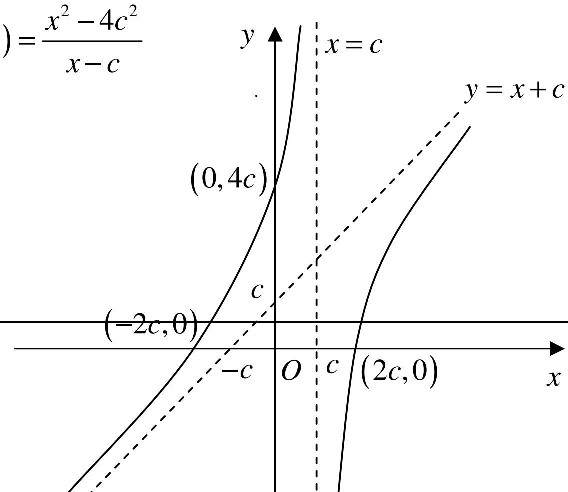
Qn	Solution
1	$\overrightarrow{OS}=\dfrac{5\mathbf{a}+4\mathbf{b}}{9}$ $\overrightarrow{MS}=\dfrac{2\mathbf{a}+4\mathbf{b}}{9}$ $\overrightarrow{MN}=\dfrac{\mathbf{b}-\mathbf{a}}{3}$ Area of $\Delta OAB=\dfrac{1}{2}\lvert\mathbf{a}\times\mathbf{b}\rvert$ Area of ΔMNS $=\dfrac{1}{2}\left\lvert\overrightarrow{MS}\times\overrightarrow{MN}\right\rvert$ $=\dfrac{1}{2}\left\lvert\dfrac{2\mathbf{a}+4\mathbf{b}}{9}\times\dfrac{\mathbf{b}-\mathbf{a}}{3}\right\rvert$ $=\dfrac{1}{2}\left(\dfrac{1}{27}\right)\left\lvert(2\mathbf{a}+4\mathbf{b})\times(\mathbf{b}-\mathbf{a})\right\rvert$ $=\dfrac{1}{2}\left(\dfrac{1}{27}\right)\left\lvert(2\mathbf{a}\times\mathbf{b})+(4\mathbf{b}\times\mathbf{b})-(2\mathbf{a}\times\mathbf{a})-(4\mathbf{b}\times\mathbf{a})\right\rvert$ $=\dfrac{1}{2}\left(\dfrac{1}{27}\right)\left\lvert(2\mathbf{a}\times\mathbf{b})+(4\mathbf{a}\times\mathbf{b})\right\rvert$ $=\dfrac{1}{2}\left(\dfrac{2}{9}\right)\lvert\mathbf{a}\times\mathbf{b}\rvert$ ratio of the area of triangle *MNS* to the area of triangle *OAB* $=\dfrac{2}{9}:1$ $=2:9$
2	

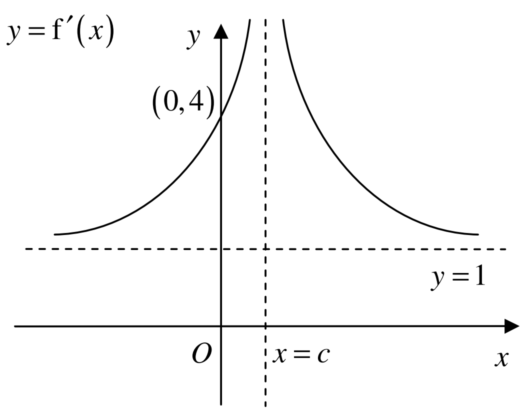
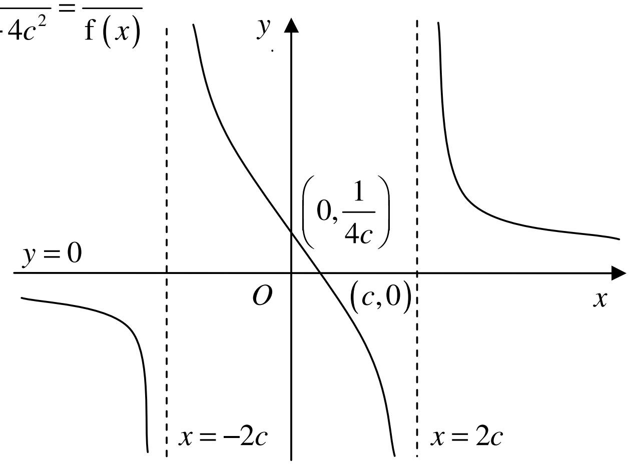
	In ΔPAB, $\cos\theta = \dfrac{AP}{AB} = \dfrac{x}{\ell}$ $\therefore$ $x = \ell\cos\theta$ $S = \dfrac{1}{2}x^2\sin\theta$ $= \dfrac{1}{2}(\ell\cos\theta)^2\sin\theta$ $= \dfrac{1}{2}\ell^2(1-\sin^2\theta)\sin\theta$ $= \dfrac{1}{2}\ell^2(\sin\theta - \sin^3\theta)$ (shown)
	$S = \dfrac{1}{2}\ell^2(\sin\theta - \sin^3\theta)$ $\dfrac{dS}{d\theta} = \dfrac{1}{2}\ell^2(\cos\theta - 3\sin^2\theta\cos\theta)$ $= \dfrac{1}{2}\ell^2\cos\theta(1-3\sin^2\theta)$ $\dfrac{dS}{d\theta} = 0 \Rightarrow \dfrac{1}{2}\ell^2\cos\theta(1-3\sin^2\theta) = 0$ $\cos\theta = 0$ (rejected) or $\sin\theta = \dfrac{1}{\sqrt{3}}$ or $\sin\theta = -\dfrac{1}{\sqrt{3}}$ (rejected) since $0 < \theta < \dfrac{\pi}{2}$ $\dfrac{dS}{d\theta} = \dfrac{1}{2}\ell^2\cos\theta(1-3\sin^2\theta)$ $\dfrac{d^2S}{d\theta^2} = \dfrac{1}{2}\ell^2\left\{(\cos\theta)(-6\sin\theta\cos\theta) + (1-3\sin^2\theta)(-\sin\theta)\right\}$ $= -\dfrac{1}{2}\ell^2\sin\theta(6\cos^2\theta + 1 - 3\sin^2\theta)$ $= -\dfrac{1}{2}\ell^2\sin\theta(9\cos^2\theta - 2)$ or $-\dfrac{1}{2}\ell^2\sin\theta(7-9\sin^2\theta)$ When $\sin\theta = \dfrac{1}{\sqrt{3}}$, $\cos^2\theta = \dfrac{2}{3}$, $\dfrac{d^2S}{d\theta^2} < 0$ $\therefore$ S is max when $\sin\theta = \dfrac{1}{\sqrt{3}}$ max $S = \dfrac{1}{2}\ell^2\left(\dfrac{1}{\sqrt{3}} - \dfrac{1}{3\sqrt{3}}\right) = \dfrac{1}{2}\ell^2\left(\dfrac{2}{3}\right)\dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{9}\ell^2$

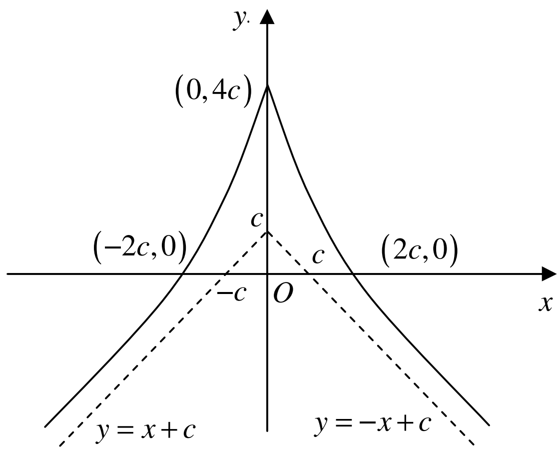
3	$\frac{dy}{dx}=\frac{6-2y}{\cos 2x}$ $\cos 2x\frac{dy}{dx}=6-2y$ *Differentiate w.r.t x* $\cos 2x\frac{d^2y}{dx^2}+\frac{dy}{dx}(-2\sin 2x)=-2\frac{dy}{dx}$ $\cos 2x\frac{d^2y}{dx^2}+\frac{dy}{dx}(2-2\sin 2x)=0$ *Differentiate w.r.t x again* $\cos 2x\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}(-2\sin 2x)+\frac{dy}{dx}(-4\cos 2x)+(2-2\sin 2x)\frac{d^2y}{dx^2}=0$ *When* $x=0,\ y=1,\ \frac{dy}{dx}=4,\frac{d^2y}{dx^2}=-8,\ \frac{d^3y}{dx^3}=32$. Using Maclaurin's theorem: $f(x)=1+4x+\frac{-8}{2!}x^2+\frac{32}{3!}x^3+.......$ $=1+4x-4x^2+\frac{16}{3}x^3+......$ $\frac{1-\sin x}{\cos x}=\left(1-x+\frac{x^3}{6}\right)\left(1-\frac{x^2}{2}\right)^{-1}$ $=\left(1-x+\frac{x^3}{6}\right)\left(1+\frac{x^2}{2}+.....\right)$ $=1-x+\frac{x^2}{2}-\frac{x^3}{3}+....$ $\frac{1-\sin x}{\cos x}=\sec x-\tan x=1-x+\frac{x^2}{2}-\frac{x^3}{3}+....$ $\therefore \tan 2x-\sec 2x=-\left(1-2x+\frac{4x^2}{2}-\frac{8x^3}{3}\right)$ Substitute power series into $f(x)=a(\tan 2x-\sec 2x)+b$ and compare coefficients of constant term and term in x $b-a=1$, $2a=4 \Rightarrow a=2$ *and* $b=3$

4 (i)	$z^3 = (-2+2\mathrm{i})$ $z^3 = \sqrt{8}\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4}\right)}$ $z^3 = 2^{\frac{3}{2}}\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4}+2k\pi\right)}$ $z = \sqrt{2}\mathrm{e}^{\mathrm{i}\pi\left(\frac{3+8k}{12}\right)}, k = 0, \pm 1$ $z = \sqrt{2}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{4}\right)}, \sqrt{2}\mathrm{e}^{\mathrm{i}\left(\frac{11\pi}{12}\right)}, \sqrt{2}\mathrm{e}^{\mathrm{i}\left(\frac{-5\pi}{12}\right)}$ Hence, remaining roots are $\sqrt{2}\mathrm{e}^{\mathrm{i}\left(\frac{11\pi}{12}\right)}$ and $\sqrt{2}\mathrm{e}^{\mathrm{i}\left(\frac{-5\pi}{12}\right)}$.
4 (ii)	Since $1+\mathrm{i}$ is a root of the equation $2w^3 + aw^2 + bw - 2 = 0$, $2(1+\mathrm{i})^3 + a(1+\mathrm{i})^2 + b(1+\mathrm{i}) - 2 = 0$ $2(-2+2\mathrm{i}) + a(2\mathrm{i}) + b(1+\mathrm{i}) - 2 = 0$ $(b-6) + (4+2a+b)\mathrm{i} = 0+0\mathrm{i}$ Comparing real terms, Comparing imaginary terms, $b - 6 = 0$ $b = 6$ $4 + 2a + b = 0$ $a = \frac{-b-4}{2}$ $\therefore a = \frac{-6-4}{2} = -5$
4 (iii)	Since polynomial equation has real coefficients, $1+\mathrm{i}$ and $1-\mathrm{i}$ are roots to the equation. $2w^3 - 5w^2 + 6w - 2 = (w-(1+\mathrm{i}))(w-(1-\mathrm{i}))(2w - A)$ Comparing constants, $-A(1+\mathrm{i})(1-\mathrm{i}) = -2$ $A(1-\mathrm{i}^2) = 2$ $A(1-(-1)) = 2$ $A = 1$ $2w^3 - 5w^2 + 6w - 2 = 0$ $(w-(1+\mathrm{i}))(w-(1-\mathrm{i}))(2w-1) = 0$ $w = 1+\mathrm{i}, \quad 1-\mathrm{i}, \quad \frac{1}{2}.$
5(i)	The vector parallel to both p_1 and p_2 $\begin{pmatrix}-2\\0\\1\end{pmatrix} \times \begin{pmatrix}2\\1\\-2\end{pmatrix} = \begin{pmatrix}-1\\-2\\-2\end{pmatrix} = \begin{pmatrix}1\\2\\2\end{pmatrix}$

5(ii)	$\begin{pmatrix}-5\\ \alpha\\ \beta\end{pmatrix}\bullet\begin{pmatrix}-2\\ 0\\ 1\end{pmatrix}=4$ $\quad\quad$ $\begin{pmatrix}-5\\ \alpha\\ \beta\end{pmatrix}\bullet\begin{pmatrix}2\\ 1\\ -2\end{pmatrix}=6$ $10+\beta=4$ $\quad\quad$ $-10+\alpha-2\beta=6$ $\beta=-6$ $\quad\quad$ $\alpha=6+10+2(-6)=4$ The vector equation of the line in which p_1 and p_2 intersect is $\mathrm{r}=\begin{pmatrix}-5\\ 4\\ -6\end{pmatrix}+\delta\begin{pmatrix}1\\ 2\\ 2\end{pmatrix}$, $\delta\in\mathbb{R}$ Vector equation of the line in which p_1 and p_2 intersect must be perpendicular to the normal of p_3 $\begin{pmatrix}1\\ 2\\ 2\end{pmatrix}\bullet\begin{pmatrix}-6\\ 4\\ \lambda\end{pmatrix}=0$ $-6+8+2\lambda=0$ $\lambda=-1$ The point (-5, 4, -6) must not be on p_3 $\mu\neq\begin{pmatrix}-5\\ 4\\ -6\end{pmatrix}\cdot\begin{pmatrix}-6\\ 4\\ 1\end{pmatrix}=52$
6	$y=\dfrac{x^2-4c^2}{x-c}=\dfrac{x^2-c^2-3c^2}{x-c}=x+c-\dfrac{3c^2}{x-c}$ Equations of asymptotes are $y=x+c,\ x=c$ When $x=0,\ y=4c$ When $y=0,\ x=\pm 2c$ $y=\mathrm{f}\left(x\right)=\dfrac{x^2-4c^2}{x-c}$



6(i)	For $x^2-4c^2=k(x-c)$ to have two distinct positive roots, set of values of k = $\{k\in\mathbb{R}: k>4c\}$
6(ii)	$y=\dfrac{4x(x+4c)}{x+c}=4\left\{\dfrac{x^2+4cx}{x+c}\right\}=4\left\{\dfrac{(x+2c)^2-4c^2}{(x+2c)-c}\right\}=4\mathrm{f}(x+2c)$ A sequence of transformations are: (1) a translation of $2c$ units in the negative x-direction, followed by (2) a stretch/scaling with scale factor 4 parallel to the y-axis. <u>OR</u> vice versa
6 (iii) (a)	$y=\mathrm{f}(x)=\dfrac{x^2-4c^2}{x-c}=x+c-\dfrac{3c^2}{x-c}\Rightarrow y=\mathrm{f}'(x)=1+\dfrac{3c^2}{(x-c)^2}$ 
6 (iii) (b)	$y=\dfrac{x-c}{x^2-4c^2}=\dfrac{1}{\mathrm{f}(x)}$ 

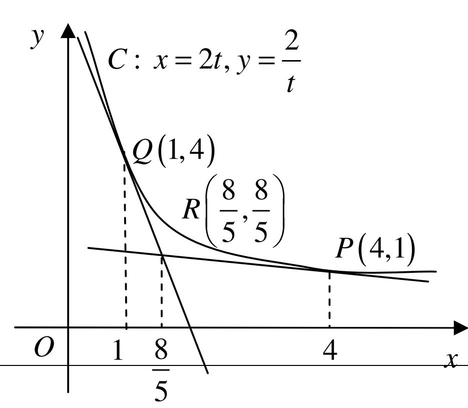
6 (iii) (c)	$y=\dfrac{4c^2-x^2}{\lvert x\rvert+c}=\dfrac{x^2-4c^2}{-\lvert x\rvert-c}=\mathrm{f}\left(-\lvert x\rvert\right)$ 
7(a)	Let P(n) be the statement $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$ for all positive integer n. When $n=1$, LHS = $\displaystyle\sum_{r=1}^{1}\frac{1}{r(r+1)(r+2)}=\frac{1}{1(1+1)(1+2)}=\frac{1}{6}$ RHS = $\dfrac{1}{4}-\dfrac{1}{2(1+1)(1+2)}=\dfrac{1}{6}$ $\therefore$ LHS = RHS $\therefore$ P(1) is true. Assume that P(k) is true for some positive integer k, i.e., $\displaystyle\sum_{r=1}^{k}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}-\frac{1}{2(k+1)(k+2)}$. To prove P($k$ + 1) is true, i.e., $\displaystyle\sum_{r=1}^{k+1}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}-\frac{1}{2(k+2)(k+3)}$, $\displaystyle\text{LHS}=\sum_{r=1}^{k+1}\frac{1}{r(r+1)(r+2)}$ $\displaystyle=\sum_{r=1}^{k}\frac{1}{r(r+1)(r+2)}+\frac{1}{(k+1)(k+2)(k+3)}$ $\displaystyle=\frac{1}{4}-\frac{1}{2(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}\left(\text{by assumption}\right)$ $\displaystyle=\frac{1}{4}-\frac{k+3-2}{2(k+1)(k+2)(k+3)}$ $\displaystyle=\frac{1}{4}-\frac{k+1}{2(k+1)(k+2)(k+3)}$ $\displaystyle=\frac{1}{4}-\frac{1}{2(k+2)(k+3)}=\text{RHS}$ Since P(1) is true and P(k) is true $\Rightarrow$ P(k+1) is true, by the Principle of Mathematical Induction, we conclude that P(1), P(2), P(3), … are all true.

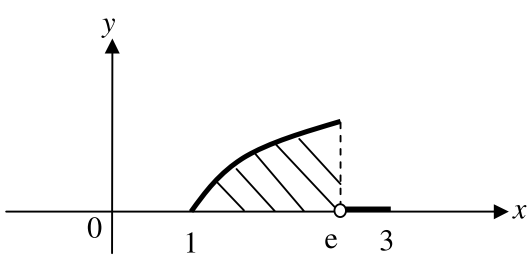
	Hence P(n) is true for all positive integers. $$\frac{5n+3}{n(n+1)(n+2)(n+3)}=\frac{a}{n(n+1)(n+2)}+\frac{b}{(n+1)(n+2)(n+3)}$$ $a=1,\ b=4$ $$\begin{aligned}&\sum_{r=1}^{n}\frac{5r+3}{r(r+1)(r+2)(r+3)}\\&=\sum_{r=1}^{n}\left(\frac{1}{r(r+1)(r+2)}+\frac{4}{(r+1)(r+2)(r+3)}\right)\\&=\sum_{r=1}^{n}\left(\frac{1}{r(r+1)(r+2)}\right)+4\sum_{r=1}^{n}\left(\frac{1}{(r+1)(r+2)(r+3)}\right)\\&=\sum_{r=1}^{n}\left(\frac{1}{r(r+1)(r+2)}\right)+4\left[\sum_{r=1}^{n+1}\left(\frac{1}{r(r+1)(r+2)}\right)-\frac{1}{1(2)(3)}\right]\\&=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}+4\left[\frac{1}{4}-\frac{1}{2(n+2)(n+3)}-\frac{1}{6}\right]\\&=\frac{7}{12}-\frac{1}{2(n+1)(n+2)}-\frac{2}{(n+2)(n+3)}\end{aligned}$$
7(b)	$$\frac{1}{n-2}-\frac{1}{n+2}=\frac{(n+2)-(n-2)}{(n+2)(n-2)}=\frac{4}{(n+2)(n-2)}=\frac{4}{n^2-4}$$

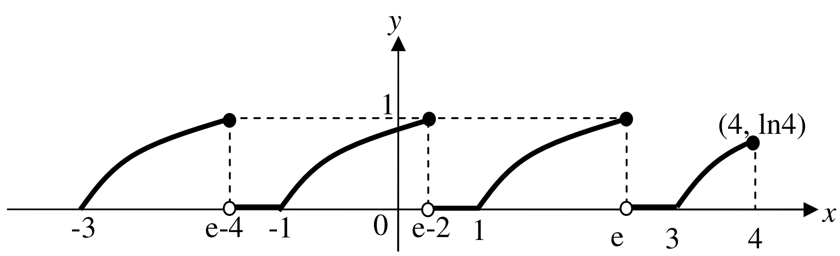
	$\sum_{r=3}^{n}\frac{1}{r^2-4}$ $=\frac{1}{4}\sum_{r=3}^{n}\frac{4}{(r-2)(r+2)}$ $=\frac{1}{4}\sum_{r=3}^{n}\left(\frac{1}{r-2}-\frac{1}{r+2}\right)$ $=\frac{1}{4}\begin{pmatrix}\frac{1}{1}-\cancel{\frac{1}{5}}\\+\frac{1}{2}-\cancel{\frac{1}{6}}\\+\frac{1}{3}-\cancel{\frac{1}{7}}\\+\frac{1}{4}-\cancel{\frac{1}{8}}\\+\cancel{\frac{1}{5}}-\cancel{\frac{1}{9}}\\+\cancel{\frac{1}{6}}-\cancel{\frac{1}{10}}\\\vdots\\+\cancel{\frac{1}{n-6}}-\cancel{\frac{1}{n-2}}\\+\cancel{\frac{1}{n-5}}-\frac{1}{n-1}\\+\cancel{\frac{1}{n-4}}-\frac{1}{n}\\+\cancel{\frac{1}{n-3}}-\frac{1}{n+1}\\+\cancel{\frac{1}{n-2}}-\frac{1}{n+2}\end{pmatrix}$ $=\frac{1}{4}\left[1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{n-1}-\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}\right]$ $=\frac{25}{48}-\frac{1}{4}\left[\frac{1}{n-1}+\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}\right]$ When $n\to\infty$, $\frac{1}{n-1}\to 0$, $\frac{1}{n}\to 0$, $\frac{1}{n+1}\to 0$, $\frac{1}{n+2}\to 0$.. Therefore the series $\sum_{r=3}^{\infty}\frac{1}{r^2-4}$ converges. $\sum_{r=3}^{\infty}\frac{1}{r^2-4}\to\frac{25}{48}$
8(i)	The acute angle between p_1 and p_2

	$= \cos^{-1}\left\lvert \dfrac{\begin{pmatrix}-7\\4\\4\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\2\end{pmatrix}}{\sqrt{7^2+4^2+4^2}\sqrt{1^2+2^2+2^2}} \right\rvert$ $= 74.97° = 75.0°$
8(ii)	$\begin{pmatrix}1\\0\\0\end{pmatrix}$ is a point on p_1. $\begin{pmatrix}1\\2\\0\end{pmatrix}$ is a point on p_2. Let point D be $\begin{pmatrix}1\\0\\0\end{pmatrix}$. Let point C be $\begin{pmatrix}1\\2\\0\end{pmatrix}$. $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$ $= \begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}2\\\alpha\\3\end{pmatrix} = \begin{pmatrix}-1\\-\alpha\\-3\end{pmatrix}$ $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$ $= \begin{pmatrix}1\\2\\0\end{pmatrix} - \begin{pmatrix}2\\\alpha\\3\end{pmatrix} = \begin{pmatrix}-1\\2-\alpha\\-3\end{pmatrix}$ Perpendicular distance from A to p_1 $\left\lvert \dfrac{\overrightarrow{AD}\cdot\begin{pmatrix}-1\\-2\\2\end{pmatrix}}{\sqrt{1^2+2^2+2^2}} \right\rvert$ $= \left\lvert \dfrac{\begin{pmatrix}-1\\-\alpha\\-3\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\2\end{pmatrix}}{\sqrt{1^2+2^2+2^2}} \right\rvert$ $= \left\lvert \dfrac{1+2\alpha-6}{\sqrt{9}} \right\rvert$ $= \left\lvert \dfrac{-5+2\alpha}{3} \right\rvert$ Perpendicular distance from A to p_2 $\left\lvert \dfrac{\overrightarrow{AC}\cdot\begin{pmatrix}-7\\4\\4\end{pmatrix}}{\sqrt{7^2+4^2+4^2}} \right\rvert$ $= \left\lvert \dfrac{\begin{pmatrix}-1\\2-\alpha\\-3\end{pmatrix}\cdot\begin{pmatrix}-7\\4\\4\end{pmatrix}}{\sqrt{7^2+4^2+4^2}} \right\rvert$ $= \left\lvert \dfrac{7+8-4\alpha-12}{\sqrt{81}} \right\rvert$ $= \left\lvert \dfrac{3-4\alpha}{9} \right\rvert$

	$\therefore \left\|\dfrac{-5+2\alpha}{3}\right\| = \left\|\dfrac{3-4\alpha}{9}\right\|$ $\dfrac{-5+2\alpha}{3} = \dfrac{3-4\alpha}{9}$ $-15+6\alpha = 3-4\alpha$ $10\alpha = 18$ $\alpha = \dfrac{18}{10} = \dfrac{9}{5} = 1\dfrac{4}{5}$ or $\dfrac{-5+2\alpha}{3} = -\dfrac{3-4\alpha}{9}$ $-15+6\alpha = -3+4\alpha$ $2\alpha = 12$ $\alpha = \dfrac{12}{2} = 6$
8(iii)	Let *M* be the position vector of the foot of perpendicular from *B* to p_1. Equation of line segment *BM* $r = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$ When line segment *BM* intersects p_1. $\begin{pmatrix} -\lambda \\ 1-2\lambda \\ 2+2\lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} = -1$ $\lambda - 2 + 4\lambda + 4 + 4\lambda = -1$ $\lambda = -\dfrac{1}{3}$ $\therefore \overrightarrow{OM} = \dfrac{1}{3}\begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$ Let point *M'* be a point in plane p_3 such that $\overrightarrow{MB} = \overrightarrow{BM'}$. $\overrightarrow{MB} = \overrightarrow{BM'}$ $\overrightarrow{OM'} = 2\overrightarrow{OB} - \overrightarrow{OM}$ $\overrightarrow{OM'} = 2\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \dfrac{1}{3}\begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} = \dfrac{1}{3}\begin{pmatrix} -1 \\ 1 \\ 8 \end{pmatrix}$ Equation of plane p_3 $r.\begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} = \dfrac{1}{3}\begin{pmatrix} -1 \\ 1 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} = 5$ $-x - 2y + 2z = 5$

9(i)	$x=2t \Rightarrow \dfrac{dx}{dt}=2$ $y=\dfrac{2}{t} \Rightarrow \dfrac{dy}{dt}=-\dfrac{2}{t^2}$ $\therefore \dfrac{dy}{dx}=-\dfrac{1}{t^2}$ Equation of tangent at $\left(2t,\dfrac{2}{t}\right)$ is $y-\dfrac{2}{t}=-\dfrac{1}{t^2}(x-2t)$ $y=-\dfrac{1}{t^2}x+\dfrac{4}{t}$
9(ii)	Since $\dfrac{dy}{dx}=-\dfrac{1}{t^2}$, as $t\to\pm\infty$, $\dfrac{dy}{dx}\to 0$ Tangents to C become parallel to the x-axis, hence Normals to C become parallel to the y-axis. <u>OR</u> normals to C become vertical lines.
9(iii)	Equation of tangent at P is $y=-\dfrac{1}{p^2}x+\dfrac{4}{p}$ Equation of tangent at Q is $y=-\dfrac{1}{q^2}x+\dfrac{4}{q}$ At R, $-\dfrac{1}{p^2}x+\dfrac{4}{p}=-\dfrac{1}{q^2}x+\dfrac{4}{q}$ $x\left(\dfrac{1}{q^2}-\dfrac{1}{p^2}\right)=\dfrac{4}{q}-\dfrac{4}{p}$ $x\left(\dfrac{p^2-q^2}{p^2q^2}\right)=4\left(\dfrac{p-q}{pq}\right)$ $x=\dfrac{4pq}{p+q}$ (shown) $y=-\dfrac{1}{p^2}\left(\dfrac{4pq}{p+q}\right)+\dfrac{4}{p}$ $\quad$ <u>OR</u> $\quad$ $y=-\dfrac{1}{q^2}\left(\dfrac{4pq}{p+q}\right)+\dfrac{4}{q}$ $=\dfrac{4}{p}\left(1-\dfrac{q}{p+q}\right)$ $\quad\quad$ $=\dfrac{4}{q}\left(1-\dfrac{p}{p+q}\right)$ $=\dfrac{4}{p+q}$ $\quad\quad$ $=\dfrac{4}{p+q}$
9(iv)	$pq=1 \Rightarrow x=\dfrac{4pq}{p+q}=\dfrac{4}{p+q}=y$ Cartesian equation of locus of R is $y=x$
9(v)	

	Required area $= \int_1^4 y\,dx - \frac{1}{2}\left(4+\frac{8}{5}\right)\left(\frac{3}{5}\right) - \frac{1}{2}\left(1+\frac{8}{5}\right)\left(\frac{12}{5}\right)$ $= \int_{\frac{1}{2}}^{2} \frac{2}{t} 2dt - \frac{42}{25} - \frac{78}{25}$ $= \left[4\ln\lvert t\rvert\right]_{\frac{1}{2}}^{2} - \frac{24}{5}$ $= 8\ln 2 - \frac{24}{5}$
10 (i)	$y = \ln x$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$ $\mathrm{d}x = x\,\mathrm{d}y$ $\mathrm{d}x = \mathrm{e}^y\,\mathrm{d}y$ $\int (\ln x)^2\,\mathrm{d}x = \int y^2\,\mathrm{d}x$ $= \int y^2\mathrm{e}^y\mathrm{d}y$ (Shown) $\int (\ln x)^2\,\mathrm{d}x = \int y^2\mathrm{e}^y\mathrm{d}y$ $= y^2\mathrm{e}^y - \int \mathrm{e}^y(2y)\,\mathrm{d}y$ $= y^2\mathrm{e}^y - 2\int \mathrm{e}^y(y)\,\mathrm{d}y$ $= y^2\mathrm{e}^y - 2\left[y\mathrm{e}^y - \int \mathrm{e}^y(1)\,\mathrm{d}y\right]$ $= y^2\mathrm{e}^y - 2\left[y\mathrm{e}^y - \mathrm{e}^y\right] + c$ $= \mathrm{e}^y\left\{y^2 - 2y + 2\right\} + c$ $= x\left\{(\ln x)^2 - 2(\ln x) + 2\right\} + c$ (*Shown*)
10 (ii) (a)	

	$\text{Volume} = \pi\int_1^{e}(\ln x)^2\,dx$ $= \pi\left[x\left\{(\ln x)^2 - 2(\ln x) + 2\right\}\right]_1^{e}$ $= \pi\left[\left(e(1)^2 - 2e + 2e\right) - (2)\right]$ $= \pi(e-2)\text{ units}^3$
10 (ii) (b)	 Method 1: $\int_{-3}^{4} f(x)\,dx = 3\int_1^{e}\ln x\,dx + \int_1^{2}\ln x\,dx$ $= 3(1) + 0.3863$ $= 3.3863$ $= 3.39$ Method 2: $\int_{-3}^{4} f(x)\,dx = 3\int_1^{e}\ln x\,dx + \int_3^{4}\ln(x-2)\,dx$ $= 3(1) + 0.3863$ $= 3.3863$ $= 3.39$ (3s.f)