**Anderson Junior College**
**Preliminary Examination 2014**
**H2 Mathematics Paper 1 (9740/01)**

**1**

$$\frac{x-5}{x-1} \le -1 \;\Rightarrow\; \frac{x-5}{x-1} + 1 \le 0$$

$$\frac{2x-6}{x-1} \le 0$$

$$(x-3)(x-1) \le 0 \;\Rightarrow 1 \le x \le 3$$

The solution is $1 < x \le 3$ since $x-1 \ne 0$

$$\frac{\cos x - 4}{\cos x} \le -1$$

$$\Rightarrow \frac{(\cos x + 1) - 5}{(\cos x + 1) - 1} \le -1.$$

The solution is $1 < \cos x + 1 \le 3$

$\Rightarrow 0 < \cos x \le 2$

$\Rightarrow 0 \le x < \dfrac{\pi}{2}$

**2**

Let the areas of the sectors be $A_1$, $A_2$, …… $A_{22}$

$A_8 = 2\,A_1$

$A_1 + 7d = 2A_1$

$A_1 = 7d$ …….. (1)

Sum of all the areas = area of the circle

$$\frac{22}{2}\left[2A_1 + 21d\right] = \pi r^2$$

$11\left[35d\right] = \pi r^2$ since $A_1 = 7d$

$$d = \frac{1}{385}\pi r^2$$

$A_{22} = A_1 + 21d = 28d$

$$\frac{1}{2}r^2\theta = 28\left(\frac{1}{385}\pi r^2\right)$$

$$\theta = \frac{8\pi}{55}$$

Alternative :

$A_1 + A_2 + \ldots\ldots + A_{22} = \pi r^2$

$7d + 8d + \ldots\ldots + 28d = \pi r^2$

$$\frac{22}{2}\left[7d + 28d\right] = \pi r^2$$

$$385d = \pi r^2$$

$$r^2 = \frac{385d}{\pi}$$

$$A_{22} = \frac{1}{2}r^2\theta = 28d$$

$$\theta = \frac{56d}{r^2} = 56d \times \frac{\pi}{385d} = \frac{8\pi}{55}$$

**3a**

$$\int \frac{6+2x}{\sqrt{1-4x-x^2}}\,dx = \int \frac{2-(-4-2x)}{\sqrt{1-4x-x^2}}\,dx$$

$$= \int \frac{2}{\sqrt{1-4x-x^2}}\,dx - \int \frac{(-4-2x)}{\sqrt{1-4x-x^2}}\,dx$$

$$= \int \frac{2}{\sqrt{5-(x+2)^2}}\,dx - \int \frac{-4-2x}{\sqrt{1-4x-x^2}}\,dx$$

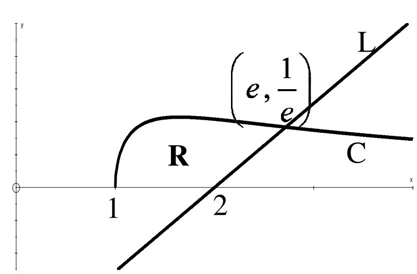
$$= 2\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) - 2\sqrt{1-4x-x^2} + c$$

**3b**

Point of intersection: $\left(e, \dfrac{1}{e}\right)$

Volume

$$= \pi\int_1^e \left(\frac{\sqrt{\ln x}}{x}\right)^2 dx - \frac{\pi}{3}\left(\frac{1}{e}\right)^2 (e-2)$$



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$$= \pi \int_1^e \frac{\ln x}{x^2}\,\mathrm{d}x - \frac{\pi(e-2)}{3e^2}$$

$$= \pi \left[ (\ln x)\left(-\frac{1}{x}\right) - \int \left(-\frac{1}{x}\right)\frac{1}{x}\,\mathrm{d}x \right]_1^e - \frac{\pi(e-2)}{3e^2}$$

$$= \pi \left[ \left(-\frac{\ln x}{x}\right) - \frac{1}{x} \right]_1^e - \frac{\pi(e-2)}{3e^2}$$

$$= \pi \left[ 1 - \frac{2}{e} \right] - \frac{\pi(e-2)}{3e^2}$$

$$= \pi - \frac{2\pi}{e} - \frac{\pi}{3e} + \frac{2\pi}{3e^2}$$

$$= \pi\left(1 - \frac{7}{3e} + \frac{2}{3e^2}\right)$$

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<tr><td>4</td><td>

$$\frac{dy}{dx} = \frac{-2x(1+a)}{\left(1+x^2\right)^2} \quad ; \quad \frac{d^2y}{dx^2} = \frac{2(1+a)\left(3x^2-1\right)}{\left(1+x^2\right)^3}$$

$$\frac{dy}{dx} = 0 \Rightarrow -2x(1+a) = 0 \Rightarrow x = 0, \quad \text{if } 1+a \neq 0$$

$$when\ x=0,\ \frac{d^2y}{dx^2} = \frac{2(1+a)(-1)}{(1)^3} < 0 \Rightarrow 1+a>0 \Rightarrow a>-1$$

$\Rightarrow\ y = \mathrm{f}(x)$ has a maximum point if $a > -1$

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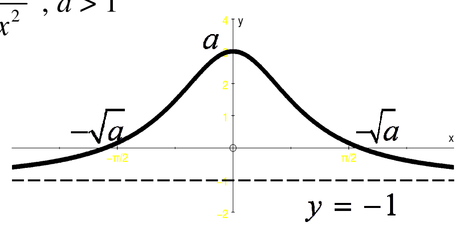
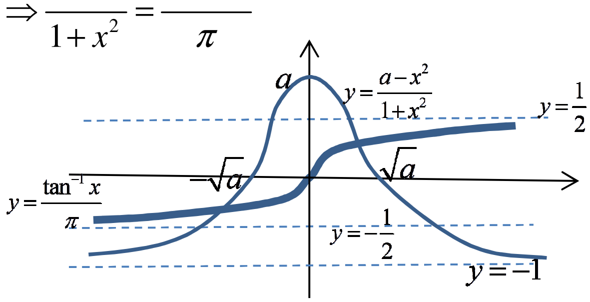
<tr><td>4(i)<br><br><br><br>4(ii)</td><td>

$$\mathrm{f}(x) = \frac{a-x^2}{1+x^2}\ , a>1$$




$$\pi\left(a-x^2\right) = \left(1+x^2\right)\tan^{-1} x$$

$$\Rightarrow \frac{a-x^2}{1+x^2} = \frac{\tan^{-1} x}{\pi}$$




There are 2 intersections of the 2 curves

$y = \dfrac{a-x^2}{1+x^2}$ and $y = \dfrac{\tan^{-1} x}{\pi}$,

hence 2 real roots of the eqn $\pi\left(a-x^2\right) = \left(1+x^2\right)\tan^{-1} x$.

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<tr><td>5(i)</td><td>

(i) Let $y = \dfrac{1}{1-x^2} \quad \Rightarrow x = \pm\sqrt{1-\dfrac{1}{y}}$

$$\Rightarrow x = -\sqrt{1-\frac{1}{y}}\ , -1 < x \leq 0$$

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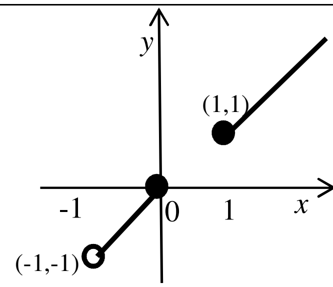
|---|---|
| | $\Rightarrow f^{-1}(x)=-\sqrt{1-\frac{1}{x}}\quad ,x\geq 1$ |
| 5(ii) | (ii) $ff^{-1}(x)=x\quad ,x\geq 1$<br>$f^{-1}f(x)=x\quad ,-1<x\leq 0$<br><br>From the graphs, there are no solutions for $ff^{-1}(x)=f^{-1}f(x)$<br> |
| 5 (iii) | (iii) $h(x)=-\frac{1}{2}-e^{-x-1}$ |
| 5 (iv) | $R_h=[-\frac{3}{2},-\frac{1}{2})$<br>$D_f=(-1,0]$<br>$R_h\not\subset D_f\Rightarrow$ fh does not exist (shown) |
| 5 (v) | For $R_h\subseteq D_f\Rightarrow R_h=(-1,-\frac{1}{2})$<br>When $x=-1$,<br>$-1=-\frac{1}{2}-e^{-x-1}$<br>$e^{x+1}=2$<br>$x=\ln 2-1$<br>Maximal domain for h is $(\ln 2-1,\infty)$<br>When $x\to -1,\ f(x)\to\infty$<br>When $x=-\frac{1}{2},\ f\left(-\frac{1}{2}\right)=\frac{1}{1-(1/4)}=\frac{4}{3}$<br>$R_{fh}=\left(\frac{4}{3},\infty\right)$ |
| 6(i) | $y=\int\frac{dy}{dx}\,dx=\int\frac{1}{2t-1}\frac{dx}{dt}\,dt=\int\frac{t-2}{2t-1}\,dt$<br>$=\int\left(\frac{1}{2}-\frac{3}{2}.\frac{1}{2t-1}\right)dt\quad=\frac{1}{2}t-\frac{3}{4}\ln(2t-1)+c$<br>When $t=4$, $y=3-\frac{3}{4}\ln 7=\frac{1}{2}(4)-\frac{3}{4}\ln 7+c\Rightarrow c=1$<br>Hence $y=\frac{1}{2}t-\frac{3}{4}\ln(2t-1)+1$ |
| 6(ii) | $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{dy}{dx}\right)\times\frac{dt}{dx}$<br>$=\frac{d}{dt}\left(\frac{1}{2t-1}\right)\cdot\frac{dt}{dx}=\frac{-2}{(2t-1)^2}\cdot\frac{1}{t-2}=\frac{-2}{(2t-1)^2(t-2)}$ |
| 6(iii) | When $x=0$, $\Rightarrow\frac{1}{2}t^2-2t=0\quad\Rightarrow t\left(\frac{t}{2}-2\right)=0\quad\Rightarrow t=0$ (rejected) or $t=4$<br>Hence when $x=0$, $y=3-\frac{3}{4}\ln 7$; $\frac{dy}{dx}=\frac{1}{2t-1}=\frac{1}{7}$; $\frac{d^2y}{dx^2}=\frac{-1}{49}$.<br>By Maclaurin's Theorem, $y=3-\frac{3}{4}\ln 7+\frac{1}{7}x-\frac{1}{98}x^2+\ldots..$ |

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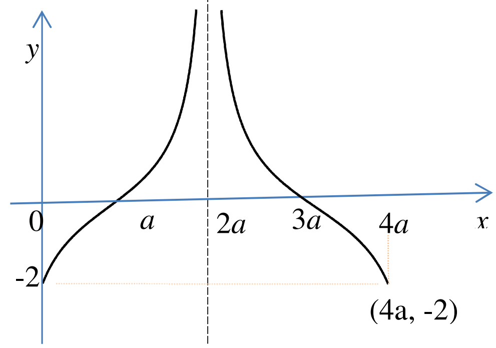
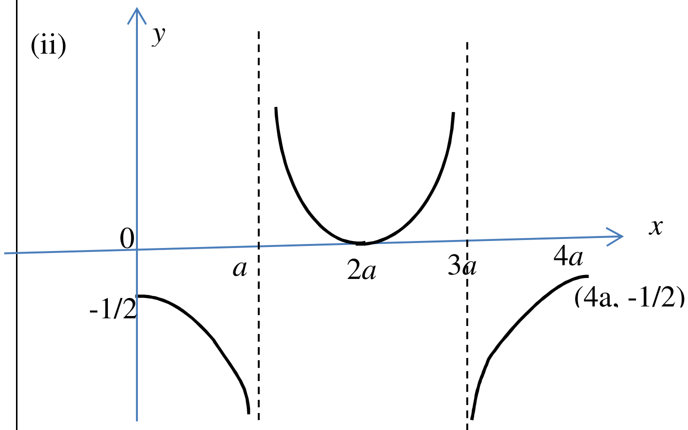
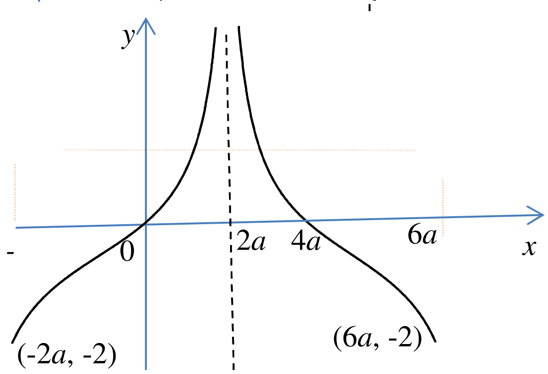
<tr><td>7</td><td>(i)




(ii)




(iii)



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<tr><td>8</td><td>Let F be the foot of perpendicular from A to the plane $P_2$.

Equation of line AF is given by $\boldsymbol{r} = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad s \in \mathbb{R}$

Since F lies on the line, $\overrightarrow{OF} = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, for some $s \in \mathbb{R}$.

Since F also lies on plane $P_2$,

$\begin{pmatrix} -3-s \\ 10 \\ 3+s \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 1 \quad \Rightarrow \quad (3+s)+(3+s) = 1 \quad \Rightarrow \quad s = -\frac{5}{2}$

Therefore, $\overrightarrow{OF} = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} - \frac{5}{2}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 10 \\ \frac{1}{2} \end{pmatrix}$

Coordinates of foot of perpendicular = $\left(-\frac{1}{2}, 10, \frac{1}{2}\right)$.

By ratio theorem, $\overrightarrow{OF} = \dfrac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$
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$$\Rightarrow \ \overrightarrow{OB} = 2\overrightarrow{OF} - \overrightarrow{OA} \quad = 2\begin{pmatrix} -\frac{1}{2} \\ 10 \\ \frac{1}{2} \end{pmatrix} - \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ -2 \end{pmatrix}$$

Hence $(2,10,-2)$ is the reflection of $(-3,10,3)$ in $P_2$ .

When the planes intersect,

$$\left(\begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 12 \\ 3 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) . \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 1$$

$\Rightarrow \quad 6 + 5\lambda = 1$

$\Rightarrow \qquad \lambda = -1$

Subst $\lambda = -1$,

$$r = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 12 \\ 3 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} .$$

$\therefore$equation of L is

$$r = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

Alternative:

$$\begin{pmatrix} -2 \\ 12 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ -14 \end{pmatrix}$$

$$P_1 : \ \mathbf{r}\bullet\begin{pmatrix} 9 \\ 5 \\ -14 \end{pmatrix} = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix}\bullet\begin{pmatrix} 9 \\ 5 \\ -14 \end{pmatrix} = -19$$

$P_1 : 9x+5y-14z = -19$

$P_2 : -x+z = 1$

By GC, $x=-1+z$

$y=-2+z$

$$L : r = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

A$(-3,10,3)$ is on $P_1$, B$(2,10,-2)$ lies on plane $P_3$.

Line L also lies on $P_3$.

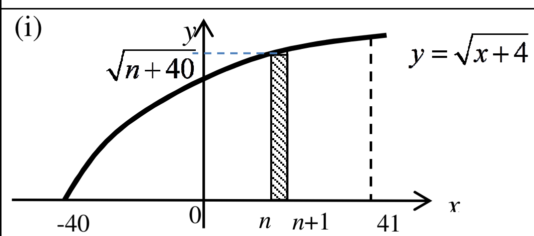
Therefore the vector $\begin{pmatrix} 2 \\ 10 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 12 \\ -2 \end{pmatrix}$ is parallel to $P_3$.

Hence a vector $\perp$ to $P_3$ is given by $\begin{pmatrix} 3 \\ 12 \\ -2 \end{pmatrix} \text{x} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix}$ .

Equation of plane $P_3$ is given by:

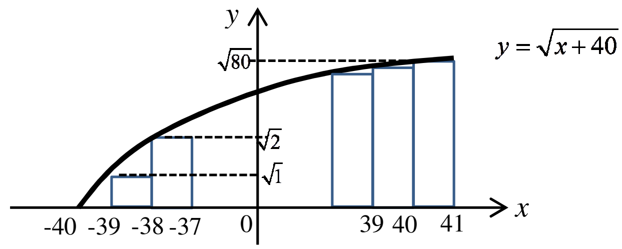
$$\mathbf{r}.\begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ -2 \end{pmatrix}.\begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix} \quad \Rightarrow \mathbf{r}.\begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix} = -4$$

9 (i)




area of shaded rectangle < Area under the curve from x= n to x = n+1

hence $\quad 1 \text{ x } \sqrt{n+40} \quad < \int_{n}^{n+1} \sqrt{x+40}\ dx$

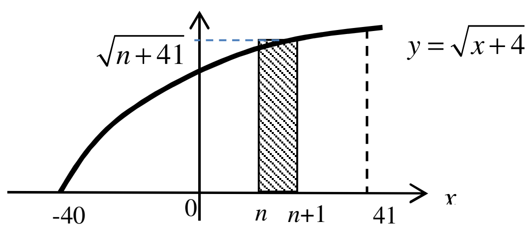



Drawing similar triangles as shown above,
Sum of area of rectangles < actual area under curve

$$\sqrt{0}\times1+\sqrt{1}\times1+\sqrt{2}\times1+....+\sqrt{80}\times1 \quad < \int_{-40}^{41}\sqrt{x+40}\ \mathrm{d}x$$

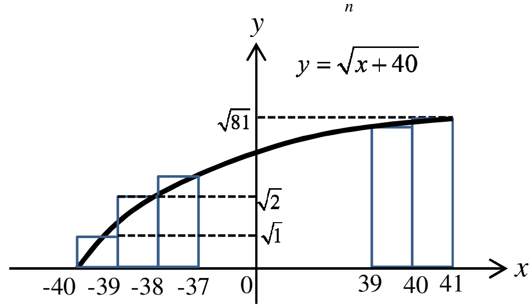
Hence $\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{80} < \int_{-40}^{41}\sqrt{x+40}\ \mathrm{d}x$ --- (1)

Draw rectangle as shown below:




area of shaded rectangle > Area under the curve from x= n to x = n+1

hence 1 x $\sqrt{n+41} > \int_{n}^{n+1}\sqrt{x+40}\ \mathrm{d}x$




By drawing rectangles as shown above,
Sum of area of rectangles > actual area under curve

$$\sqrt{1}\times1+\sqrt{2}\times1+....+\sqrt{81}\times1 > \int_{-40}^{41}\sqrt{x+40}\ \mathrm{d}x$$

Hence $\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{81} > \int_{-40}^{41}\sqrt{x+40}\ \mathrm{d}x$ --- (2)

$$\int_{-40}^{41}\sqrt{x+40}\ \mathrm{d}x = \left[\frac{2}{3}(x+40)^{\frac{3}{2}}\right]_{-40}^{41} \quad \left[\frac{2}{3}(x+40)^{\frac{3}{2}}\right]_{-40}^{41} = 486 \text{ (or use GC)}$$

From (1) and (2),

$$\int_{-40}^{41}\sqrt{x+40}\ \mathrm{d}x-\sqrt{81} < \sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{80} < \int_{-40}^{41}\sqrt{x+40}\ \mathrm{d}x$$

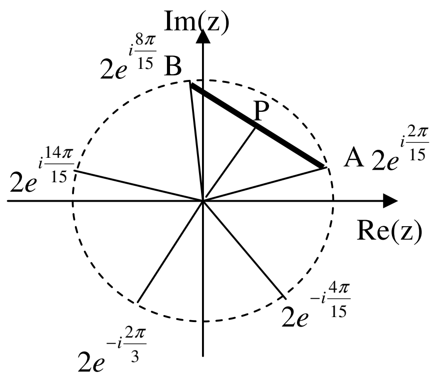
$$477<\sqrt{1}+\sqrt{2}+\sqrt{3}+..+\sqrt{80}<486$$
$$9(53)<\sqrt{1}+\sqrt{2}+\sqrt{3}+..+\sqrt{80}<9(54)$$
$$a=53$$

**10**

$$z^5+16=16\sqrt{3}i \;\Rightarrow z^5=16\left(-1+\sqrt{3}i\right)=32e^{i\frac{2\pi}{3}}$$

$$z=2e^{i\left(\frac{2k\pi+\frac{2\pi}{3}}{5}\right)},k=0,\pm1,\pm2$$

$$=2e^{i\frac{2\pi}{15}},2e^{\frac{8\pi}{15}},2e^{-i\frac{4\pi}{15}},2e^{i\frac{14\pi}{15}},2e^{-i\frac{2\pi}{3}}$$




(i) $\arg(w)=\frac{1}{2}\left(\frac{2\pi}{15}+\frac{8\pi}{15}\right)=\frac{5\pi}{15}=\frac{\pi}{3}$

$$|w|=2\cos\left(\frac{1}{2}\left(\frac{8\pi}{15}-\frac{2\pi}{15}\right)\right)=2\cos\frac{\pi}{5}$$

$(ii)w=k^{\frac{1}{n}}\Rightarrow k=w^n$

$$w=2\cos\frac{\pi}{5}e^{i\frac{\pi}{3}}$$

$$\Rightarrow w^n=\left(2\cos\frac{\pi}{5}\right)^n e^{i\frac{n\pi}{3}}=\left(2\cos\frac{\pi}{5}\right)^n\left(\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}\right)$$

$w^n$ is real if $\sin\frac{n\pi}{3}=0\Rightarrow$ least positive integer n = 3

$$k=w^3=\left(2\cos\frac{\pi}{5}\right)^3 e^{i\frac{3\pi}{3}}=-8\cos^3\frac{\pi}{5}$$

**11 (a)**

Since $u_n-u_{n-1}=\frac{e}{2^{n-1}} \;\Rightarrow \sum_{n=2}^{N}\left(u_n-u_{n-1}\right)=\sum_{n=2}^{N}\left(\frac{e}{2^{n-1}}\right)$

$$u_2-u_1=\frac{e}{2^1}$$

$$+\quad u_3-u_2=\frac{e}{2^2}$$

$$+\quad u_4-u_3=\frac{e}{2^3}$$

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$$+\quad u_N-u_{N-1}=\frac{e}{2^{N-1}}$$

$$\Rightarrow u_N-u_1=e\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^{N-1}}\right)$$

| | |
|---|---|
| | $\Rightarrow u_N - u_1 = e\left(\dfrac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{N-1}\right)}{1-\left(\frac{1}{2}\right)}\right) = e\left(1-\left(\frac{1}{2}\right)^{N-1}\right)$<br>$\Rightarrow u_N = e\left(1-\left(\frac{1}{2}\right)^{N-1}\right)+\frac{e}{10} \Rightarrow u_N = \frac{11e}{10} - e\left(\frac{1}{2}\right)^{N-1} \quad \Rightarrow u_N = \frac{e}{10}\left[11-10\left(\frac{1}{2}\right)^{N-1}\right]$<br>$u_n$ is a convergent sequence as $\left(\frac{1}{2}\right)^{N-1} \to 0$ as $N \to \infty$ |
| 11<br>(b)<br>(i) | $b_1 = \frac{1}{4}[a_1] = \frac{1}{4}$;<br>$b_2 = \frac{1}{4}[a_1+a_2] = \frac{1}{4}\left[\frac{1}{1}+\frac{1}{1+2}\right] = \frac{1}{3} = \frac{2}{6}$<br>$b_3 = \frac{1}{4}[a_1+a_2+a_3] = \frac{1}{4}\left[\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}\right] = \frac{3}{8}$<br>$b_n = \frac{n}{2n+2}$ |
| 11<br>(b)<br>(ii) | Let $P_n$ be the statement $b_n = \frac{n}{2n+2}$<br>When $n = 1$, LHS $= b_1 = \frac{1}{4}$, RHS $= \frac{1}{4}$<br>$\therefore P_1$ is true.<br>Assume that $P_k$ true for some $k \in \mathbb{Z}^+$, $k \geq 1$. i.e. Assume $b_k = \frac{k}{2k+2}$<br>For $n = k+1$, we want to prove $b_{k+1} = \frac{k+1}{2k+4}$<br>LHS $= \frac{1}{4}\left(\sum_{r=1}^{k+1} a_r\right) = \frac{1}{4}\left(\sum_{r=1}^{k} a_r\right) + \frac{1}{4}a_{k+1}$<br>$= b_k + \frac{1}{4}\left[\frac{1}{1+2+3+\ldots+(k+1)}\right]$<br>$= \frac{k}{2k+2} + \frac{1}{4}\left[\frac{1}{\frac{(k+1)(k+2)}{2}}\right]$<br>$= \frac{k}{2k+2} + \frac{1}{2(k+1)(k+2)}$<br>$= \frac{k(k+2)+1}{2(k+1)(k+2)}$<br>$= \frac{(k+1)^2}{2(k+1)(k+2)} \qquad = \frac{(k+1)}{2k+4}$ = RHS<br>$\therefore P_k$ is true $\Rightarrow P_{k+1}$ is true<br>Since $P_1$ is true and $P_k$ is true $\Rightarrow P_{k+1}$ is true, by the principle of Mathematical Induction, $P_n$ is true for all $n \in \mathbb{Z}^+$. |