
# 1 Function



Functions play a vital role in mathematics. It expresses the relationship between two quantities. We often come across functions in everyday life. For example, when we measure the distance we travel over time, we are essentially using a function to describe the relationship between distance and time. Functions also help us make predictions and solve problems, such as calculating the future value of an investment based on certain variables. Understanding functions allows us to analyze patterns and make informed decisions in a variety of fields, including economics, physics, and computer science.

**In this chapter, you will be able to:**

- understand the concepts of function, domain and range
- use of notations such as $\mathrm{f}(x) = x^2 + 5$, $\mathrm{f} : x \mapsto x^2 + 5$, $\mathrm{f}^{-1}(x)$, $\mathrm{fg}(x)$ and $\mathrm{f}^2(x)$
- find the inverse functions and composite functions
- know the conditions for the existence of inverse functions and composite functions
- restrict the domain to obtain an inverse function
- find the relationship between a function and its inverse

# 1.1 The Idea of a Function

Many real life situations involve pairs of quantities that are related to each other by some rule of correspondence. The mathematical term for such a rule of correspondence is a **relation**.

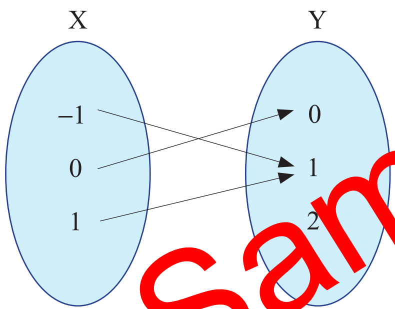
Here are the two examples.

**(a)** The area $y$ of a square is related to its side $x$ by an equation $y = x^2$.

**(b)** The time $t$ is related to $k$ kilometres at $x$ kilometres per hour by an equation $t = \frac{k}{x}$.

In example **(a)**, suppose each value of $x$ gets a unique (each) value of $y$ from it, then such a relation is called a **function**. To assist you in understand this better, let us interpret this function using a venn diagram.

The diagram belows a venn diagram which represents the mapping the set X to the set Y. Suppose the elements of the set $\mathrm{X} = \{-1, 0, 1\}$ and the elements of the set $\mathrm{Y} = \{0, 1, 2\}$ and the relation from $x$ to $y$ is given as $y = x^2$.
We have






> **Information**
> When we define a function, it is required to state the relation and *include the domain* of the function such as the one shown in (1).

and when

$$-1 \mapsto (-1)^2 = 1$$

$$0 \mapsto (0)^2 = 0$$

$$1 \mapsto (1)^2 = 1$$

From the above, we can see that each element of X gives only one element of Y, we say $y$ **is a function of** $x$.

We can use a notation to describe the function as shown below.

$$\mathrm{f} : x \mapsto x^2, \quad x \in \{-1, 0, 1\} \text{ ............................................................. (1)}$$

which means 'f is a function which turns the value of $x$ in the domain into the value of function $x^2$'.

The notation which is used for this is $\mathrm{f}(x)$ which is read as 'f $x$'. The letter f refers to the function, and $x$ for the number in the domain to be chosen for evaluation. For example, $\mathrm{f}(1)$ is the value of function when $x = 1$.

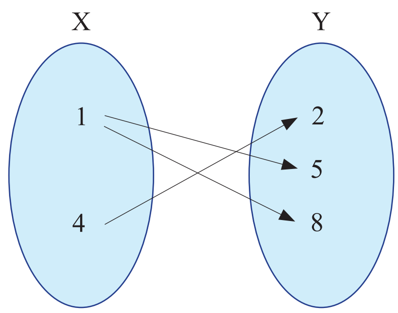
The collection of all the elements in X, i.e. $\{-1, 0, 1\}$, is called the **domain of the function**.

The collection of all the elements in Y i.e. $\{0, 1, 2\}$, is called co domain of the function.

The elements in Y, i.e. $\{0, 1\}$ corresponding to $x$ is called the **range of the function**.

Not all mappings are functions, one such example is illustrated below.

The venn diagram below shows another mapping between $x$ and $y$ with $X = \{1, 4\}$ and $Y = \{2, 5, 8\}$.




From the above, we can see that 1 gives 5 and 8. As such, this relation is not a function.

Summing up the above discussion, we can define a function as follows.



Suppose X and Y are two sets of elements. A function $f : X \rightarrow Y$ is a relation such that each element of set X (domain of the function, denoted by $D_f$) there is exactly one definite element of set Y (the range of the function, denoted by $R_f$).

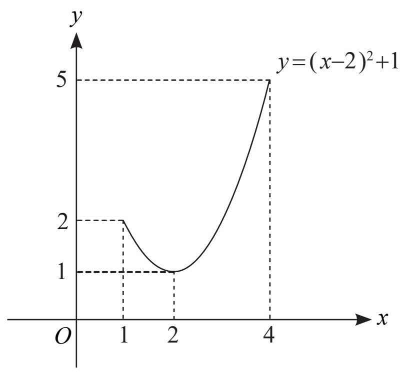
## 1.2 Graphical representation of Functions

In this section, we shall look at how a function can be represented by carteisan coordinate plane.

If function f with domain X and range Y, then its **graph** is the set of ordered pairs $(x,\ f(x))$, where $x$ is the set of values in the domain X and $y$ is the set of values in the range Y. The graph of f consists of all points $(x, y)$ in the coordinate plane such that $y = f(x)$.

For example,

$f: x \mapsto (x-2)^2+1$, with domain $\{x: 1 \le x \le 4, x \in \mathbb{R}\}$ and its graph is shown.




The graph of function f provides us a useful picture of the behaviour of a function:

Reflection

Are all equations can be described as function?

First, it tells us where the function starts and where it ends.

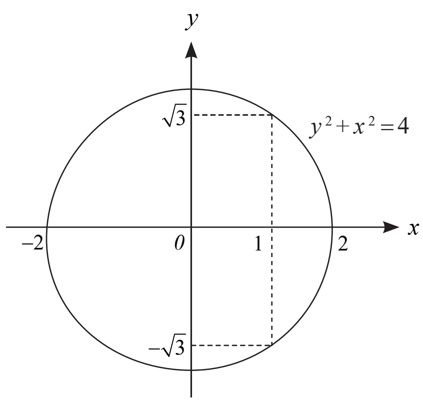
Second, it allows us to picture the domain of f on the $x$-axis and its range on the $y$-axis.
In this example, the domain of the function f is $\{x: 1 \le x \le 4, x \in \mathbb{R}\}$ while the range of the function f is $\{y: 1 \le y \le 5, y \in \mathbb{R}\}$.

Last and most importantly, it shows that for each value of $x$, it gives exactly one value of $y$ on the graph.
For example, when $x = 1$, it gives $y = 2$,
when $x = 2$, it gives $y = 1$ and so on.



However, not all graphs represent functions, one such example is illustrated below.

Consider the equation of a circle $y^2 + x^2 = 4$.




Taking the value $x = 1$, it gives two values of $y$, i.e. $y = \sqrt{3}$ and $y = -\sqrt{3}$.
From the definition of the function, we say that the graph of $y^2 + x^2 = 4$ is not a function.

## Testing Functions

In this section, we will learn to determine which curves in the coordinate plane are graphs of functions using the **Vertical Line Test**.

Let us consider the following two situations.

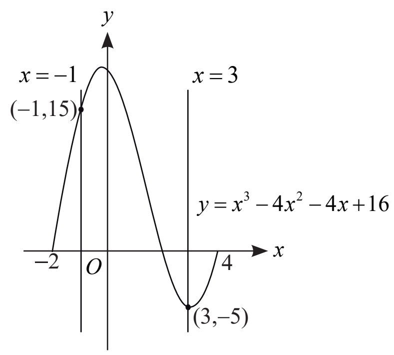
**Situation 1**

Given $y = x^3 - 4x^2 - 4x + 16$, where $-2 \le x \le 4$.

If some vertical lines are drawn within the domain of the graph, as shown. We can see that

when $x = -1$, it intersects the curve at a point, at $(-1, 15)$.

when $x = 3$, it intersects the curve at a point, at $(3, -5)$.




So, for any vertical line, $x = a$, where $a$ is the set of values of in the interval of $-2 \le x \le 4$, it intersects the curve exactly at a point.

We say the graph of $y = x^3 - 4x^2 - 4x + 16$ is a function because from the definition of a function, it states that for each value of $x$, it gives a definite value of $y$.



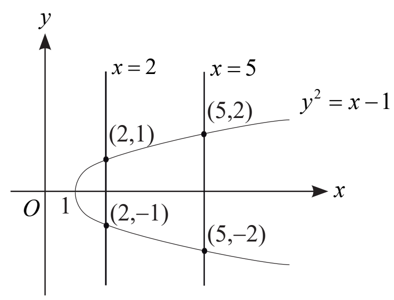
**Situation 2**

Given $y^2 = x - 1$, where $1 \le x \le 7$.

However, in this case, when some vertical lines are drawn within the domain of the graph, we can see that

when $x = 2$, it intersects the curve at (2, 1) and $(2 - 1)$.

when $x = 5$, it intersects the curve at (5, 2) and $(5 - 2)$.




So, if any vertical line, $x = a$, where $a$ is the set of values of in the interval of $1 \le x \le 7$, it intersects the curve more than once.

We say the graph of $y^2 = x - 1$ is not a function since a function cannot assign two different values to one value of $x$.

In conclusion,

If $y = f(x)$, where $x \in$ domain X and $y \in$ Range Y.
Recall the definition of the function, it states that each $x$, where $x \in$ X, there is exactly one value of $y$, where $y \in$ Y such that $y = f(x)$. It follows, then, that a vertical line intersects the graph of a function $y = f(x)$ at only point.

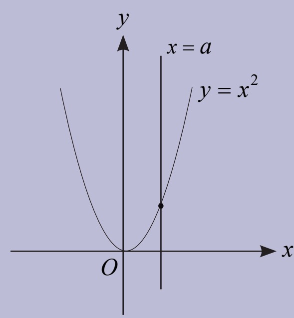
This leads to the **Vertical Line Test** for functions.

## Example 1 testing for functions

From the graph of each of the following relation, determine whether the relation is a function. State your reason(s).

### Solution

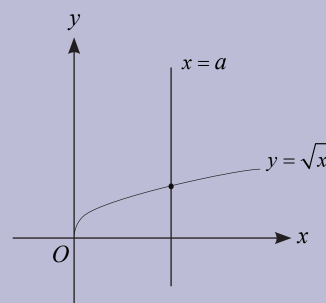
**(a)** $y = x^2,\ x \in \mathbb{R}$.




From the diagram, we can see that for any vertical line $x = a$, where $a \in \mathbb{R}$ cuts the graph of $y = x^2$ at **exactly** one point.

Thus, the equation $y = x^2$ is a function.

**(b)** $y = \sqrt{x},\ x \geq 0$.

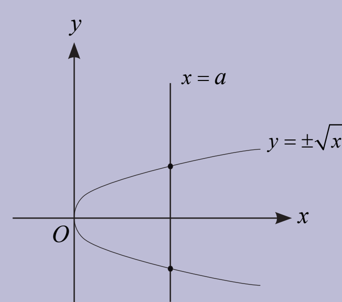



From the diagram, we can see that for any vertical line $x = a$, where $a \geq 0$ cuts the graph of $y = \sqrt{x}$ at **exactly** one point.

Thus, the equation $y = \sqrt{x}$ is a function.



**(c)** $y = \pm\sqrt{x},\ x \in \mathbb{R}^+$.




From the diagram, we can see that for any vertical line $x = a$, where $a \in \mathbb{R}^+$ cuts the graph of $y = \pm\sqrt{x}$ at two points.

Thus, the equation $y = \pm\sqrt{x}$ is not a function.

### FOLLOW UP 1

State and justify which of the following is a function.

**(a)** $f : x \to -\sqrt{x}, x \geq 0$ **(b)** $g : x \to \dfrac{1}{x}, x \in \mathbb{R}$

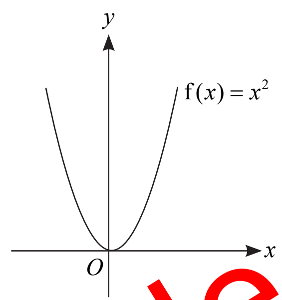
# 1.3 The domain and the range of a Function

When we deal with functions, we often encounter terms such as the domain and range. In this section, we shall take a closer look at the concept of these two terms.

## Maxima domain of a Function

We have learnt in the last section that at the set of numbers $x$ for which a function $\mathrm{f}(x)$ is defined is called the domain of the function.

Consider the function $\mathrm{f}(x) = x^2$, where $x \in \mathbb{R}$.
From the diagram, we can see that the domain for function f is $x \in \mathbb{R}$.



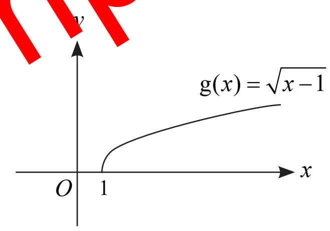

Not all functions may be defined for all real numbers.
Let us look at the following two situations.

> **Information**
>
> We may replace the letter f in the notation f($x$) with other letters such as g($x$), h($x$), and k($x$) to represent different functions of $x$.
> For example,
> $\mathrm{g}(x) \mapsto x^2 - 1, x \in \mathbb{R}$
> $\mathrm{h}(x) \mapsto x, x \in \mathbb{R}$
> $\mathrm{k}(x) \mapsto x^3, x > 1$



**Situation 1**

The diagram shows the function $\mathrm{g}(x) = \sqrt{x-1}$.
Since a negative number does not have a real square root,
$\sqrt{x-1}$ exists only if $x - 1 \geq 0$ i.e. $x \geq 1$.



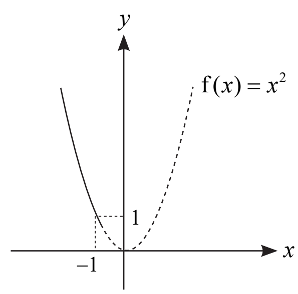

$\therefore$ the maxima domain for function g is $x \geq 1$.

**Situation 2**

Consider again the function $\mathrm{f}(x) = x^2$, $x \in \mathbb{R}$
If we restrict the domain of this function to $x < -1$, a new function is formed.

i.e. $\mathrm{h}(x) = x^2, x \in \mathbb{R}$ and $x < -1$.

$\therefore$ the maxima domain for function h is $x \in \mathbb{R}$ and $x < -1$.



## Example 2 finding the maxima domain of a function

Find the largest possible domain for each of the following functions.

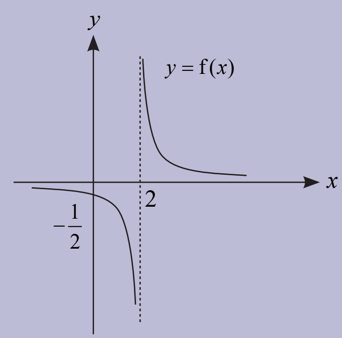
**(a)** $f(x) = \dfrac{1}{x-2}$ **(b)** $g(x) = \sqrt{x+1}$

### Solution

**(a)** The function $f(x) = \dfrac{1}{x-2}$ is undefined when $x = 2$.

This is because any number divides with 0 is undefined.

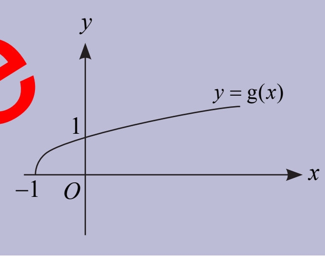
$\therefore$ the largest possible domain for function f is $D_f = x \in \mathbb{R} \setminus \{2\}$.




**(b)** The function $g(x) = \sqrt{x+1}$ only exist if $x+1 \geq 0$.

This is because the square root of a negative number is not defined.

$\therefore$ the largest possible domain for function g is $D_g = [-1, \infty)$.






**FOLLOW UP 2**

State the maximal domain for which each of the following functions exists.

**(a)** $f : x \to \ln(x-1)$ **(b)** $g : x \to 2 + \sqrt{3-x}$ **(c)** $h : x \to \dfrac{1}{x}$

## Example 3 finding the maxima domain of a function

Restrict the domain of the function $f : x \mapsto (x-1)^3, x \in \mathbb{R}$ so that the function is always positive for all real values of $x$, and write down the new function $g(x)$ and its domain.

### Solution

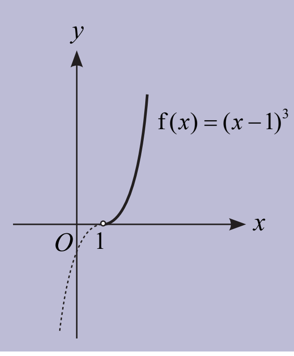
For the function to be positive, i.e. $f(x) > 0$.

We need to remove the part of the graph, as shown on the right.

(i.e. the dotted part)

Thus, the remaining domain is $x > 1$.

$\therefore$ $g : x \mapsto (x-1)^3, x \in \mathbb{R}$ and $x > 1$.



***Commentary*** :

In **Example 3**, we notice that the functions g and f give the same relation, i.e. $(x-1)^3$, but different domains. Thus, we regard f and g as two different functions.

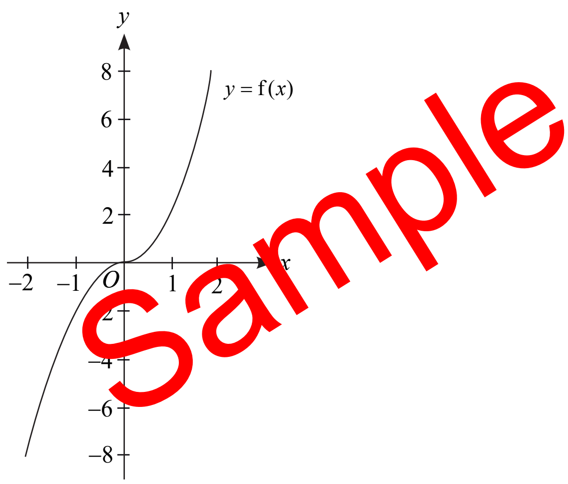
**FOLLOW UP 3**

Restrict the domain of the function $f : x \mapsto (x+1)^3, x \in \mathbb{R}$ so that the function is always positive for all real values of $x$, and write down the new function $g(x)$ and its domain.

## Range of a Function

In this section we shall learn about the range of a function.

Consider the graph of the function $f : x \mapsto x^3, x \in \mathbb{R}$ and $-2 \le x \le 2,$ as shown.




**Information**

Range of function f is a set of values from the least to the greatest values of f($x$), where $x$ is subset to the domain of f.

Recall the definition of the function, it states that each value of $x$, where $x \in$ the domain, there is exactly one value of $y$, where $y \in$ the range of the function such that $y = f(x)$.

In the diagram, we see that the set values of $y$ from $-8$ to 8 resulted from the set of values of $x$.
Thus, the range of the function of f is $-8 \le f(x) \le 8$.

We can generalise the above idea and define the range of a function as follows.

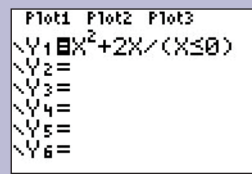
The range of the function f refers to the set of values of $y$ such that $y = f(x)$, can take from the numbers $x$, where $x \in$ domain of f.

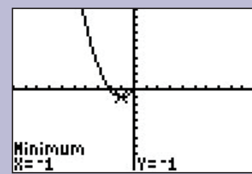
## Example 4 finding the range of a function

Find the range of the function $f: x \mapsto x^2 + 2x,\ \ x \in \mathbb{R},\ x \le 0$.

### Solution

First, we use a graphic calculator to graph the function f.
The diagram shows the graph of $f: x \mapsto x^2 + 2x, x \in \mathbb{R},\ x \le 0$.

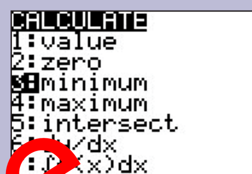


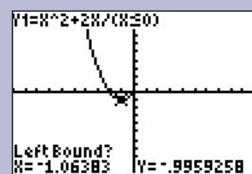




From the diagram, we observe that minimum value of the function is $-1$ and the range of the function only exists when $y \ge -1$.
$\therefore$ the range of the function f is $[-1, \infty)$.

**Footnote : Steps to determine the minimum point of the function**

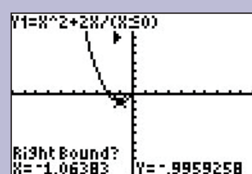
***Step 1*** : **2nd** **TRACE** under submenu '**CALCULATE**'
Scroll to 3(**minimum**) then press **ENTER** or press **3** directly.

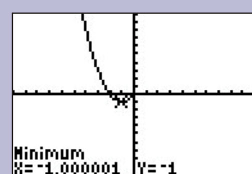



***Step 2*** : Now move the blinking cursor anywhere near to the left side of minimum point.
Press **ENTER** to select the Left Bound as shown.

***Step 3*** : Move the blinking cursor near to the right side of the same point.
Press **ENTER** to select the Right Bound.




***Step 4*** : Press **ENTER** twice. The coordinates of the minimum point is displayed.



### FOLLOW UP 4

Find the range of the function $f: x \mapsto x^2 - x - 2, x \in \mathbb{R}, -1 \le x \le 4$.

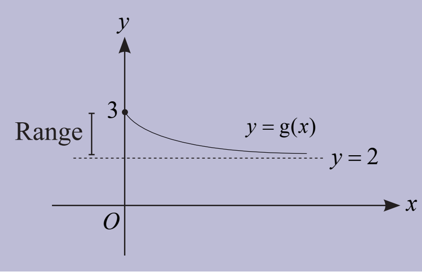
## Example 5 finding the range of a function

Find the range of the function $g : x \mapsto 2 + e^{-x}$, $x \in \mathbb{R}$ and $x \geq 0$.

### Solution

From the diagram, we notice that the function g only exists when $y > 2$ and the maximum value of the function is 3.

$\therefore$ the range of the function g is $R_g = (2, 3]$.




Alternatively, we can write the range of g as

$$\{y : 2 < y \leq 3, y \in \mathbb{R}\}.$$

**FOLLOW UP 5**

Find the range of the following.

**(a)** $f : x \to 2x + 3, -1 < x < 4$ **(b)** $g : x \to \dfrac{2x-3}{1-x}, x \in \mathbb{R} \setminus \{1\}$ **(c)** $h : x \to \sin x, x \in [0, \pi]$

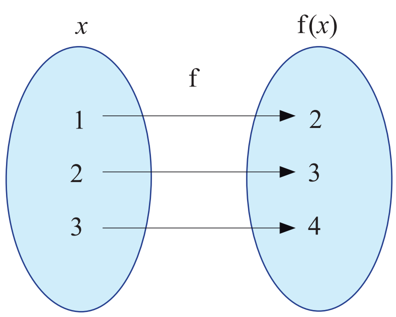


## 1.4 One-one Functions

Consider the following function

$$f : x \mapsto x + 1, \text{with domain } x = \{1, 2, 3\} \text{ and the range } f(x) = \{2, 3, 4\}.$$

The function is represented by a venn diagram as shown.

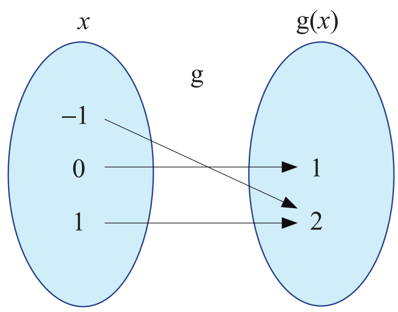



From the above, we can see that for each value of $y$ in the range of f, there is only one value of $x$, where $x$ is the domain of f such that $y = f(x)$. We say that the function f is a one-one function.

Not all functions are one-one. Let us illustrate this with an example.

Consider the following function

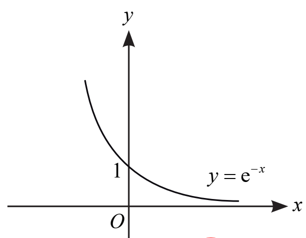
$$g : x \mapsto x^2 + 1, \text{with domain } x = \{-1, 0, 1\} \text{ and the range } g(x) = \{1, 2\}.$$



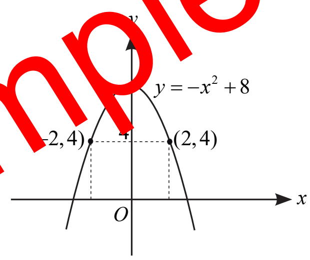

From the above illustration, we notice that there is one value of g, from two values of $x$ which is $-1$ and 1 as shown. Thus, we say that the function g is not a one-one function.

To help visualise this definition, we may view a one-one function graphically.

Consider the function $y = e^{-x}$, $x \in \mathbb{R}$.
The diagram on the right shows that for each value of $y$, it gives a unique (each) value of $x$.
We say that $y$ is a one-one function of $x$.




However, the function $y = -x^2 + 8$, $x \in \mathbb{R}$ is not a one-one function. This is because for each value of $y$, except $y = 8$, it gives two distinct values of $x$.
For example, when $y = 4$, it gives $x = 2$ and $x = -2$.






In conclusion, we define one-one function as follows.

> A function f defined for some domain D is one-one (or injective) if, each value of $y$ in the range of f there is only **one value** $x \in D$ such that $y = f(x)$.

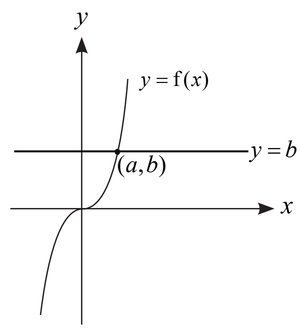
## Testing one-one Functions

We learnt to use vertical line test to determine if the curve is a function. In this section, we shall introduce another test to determine if the graph of a function is a one-one function. This test is called the **Horizontal Line** test.

Let us consider the following two situations.

**Situation 1**

The diagram shows the graph of the function $f : x \mapsto x^3, x \in \mathbb{R}$.

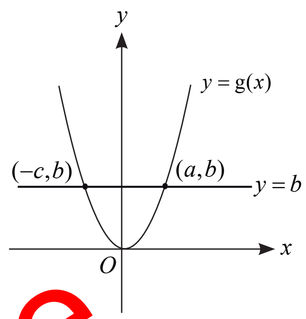



Suppose a horizontal line $y = b$, where $b$ is the subset of the range of function f, intersects the curve at exactly one point, at $(a, b)$, as shown on the right.

We say that f is a one-one function.
This is because for one-one function to exist, for one value of $y$ there is one value of $x$.

**Situation 2**

The diagram shows the graph of the function $g : x \mapsto x^2, x \in \mathbb{R}$.




Suppose a horizontal line $y = b$, where $b \in R_g$ intersects at two points at $(-c, b)$ and $(a, b)$, as shown on the right.

The function g is not a one-one function.
This is because for each of value of $y$, except $y = 0$, it results in two values of $x$, which does not satisfy the definition of 'one-one' function.

In conclusion,

> If $y = f(x)$, where $x \in$ domain X and $y \in$ Range Y.
> Recall the definition of one-one function, it states that for each $y$, where $y \in$ Y, there is exactly one value of $x$, where $x \in$ X. It follows, then, that the horizontal line $y = b$, where $b \in$ Range Y intersects the graph of the function $y = f(x)$ at only point.
> This leads to the **Horizontal Line Test** for one-one function.



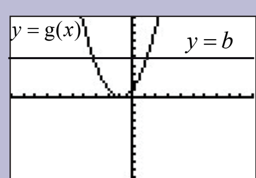
## Example 6 testing for one-one functions

Determine which of the following functions, with the specified domains, are one-one (injective/injection).

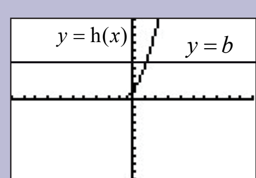
**(a)** $g : x \mapsto (x+1)^2,\ x \in \mathbb{R}$ **(b)** $h : x \mapsto (x+1)^2,\ x \in \mathbb{R},\ x \geq -1$.

### Solution

**(a)** The horizontal line, $y = b$, $b \in R_g$ cuts the graph of g at more than one point.
Hence, the function g is not one-one.




**(b)** The horizontal line $y = b$, $b \in R_h$ cuts the graph of h at **exactly** one point.
Hence, the function h is one-one.




**Reflection**

Referring to **Example 6(a)**, can the domain of g be restricted so that it becomes a one-one function?

**FOLLOW UP 6**

Determine whether each of the following functions is one-one:

**(a)** $\mathrm{f}: x \mapsto \ln x - x, x \in \mathbb{R}, 1 < x \le \mathrm{e}^2$.

**(b)** $\mathrm{h}: x \mapsto (x+1)^2, x \in \mathbb{R}, x \ge -4$.

## Example 7 restricting the domain of a function

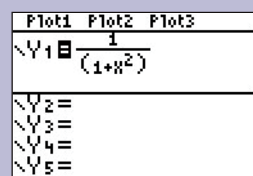
The function is defined by $\mathrm{f}: x \mapsto \dfrac{1}{1+x^2}, x \in \mathbb{R}$.

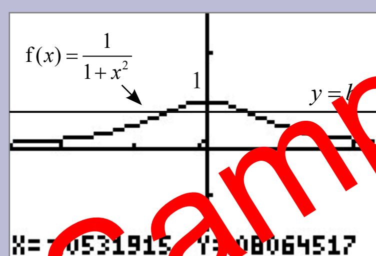
**(a)** Explain why the function f is not one-one.

**(b)** By restricting the domain of the function f, find the domain such that f is a one-one function.

### Solution

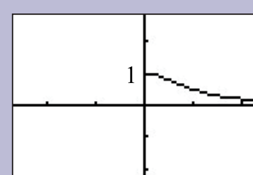
**(a)** Enter the expression and press enter to obtain the graph of the function f, as shown.



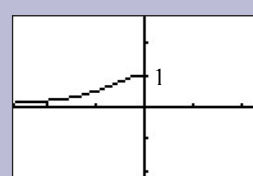



From the above diagram, we observe that any horizontal line $y = b$, where $b$ is the subset of the range of f, i.e. $0 < \mathrm{f}(x) < 1$ cuts the graph $y = \mathrm{f}(x)$ at 2 distinct points. In other words, there are two values of $x$ for each value of $y$.




Hence, f is not a one-one function.

**(b)** In order to turn function f into one-one, we need to remove part of the curve to the right (or left) of its axis of symmetry such that for every horizontal line $y = b$, where $b$ is a subset of the range of f, cuts the curve at exactly one point, i.e. resulting one value of $x$ for each value of $y$, as shown in the diagrams.




There are *two possible solutions* for the restricted domain of the function such that it is one-one function.

$\therefore$ the new domain of f for which it is one-one function is $D_f = (-\infty, 0]$ or $D_f = [0, \infty)$.

**FOLLOW UP 7**

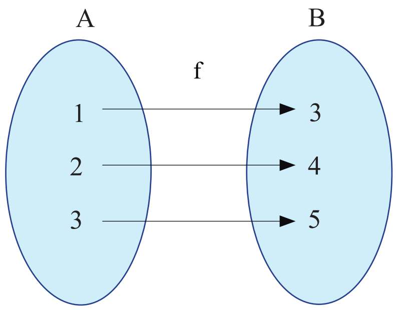
The function is defined by $\mathrm{f}: x \mapsto 1 + \mathrm{e}^{-x^2}, x \in \mathbb{R}$.

**(a)** Show that f is not one-one.

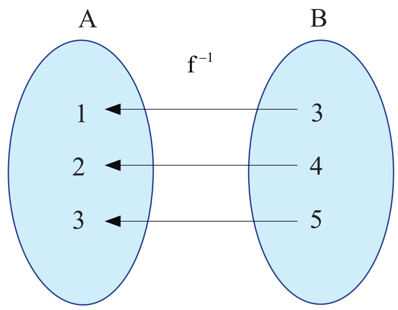
**(b)** The domain of f is further restricted to $x \le k$, state the largest value of $k$ which f is a one-one function.

## 1.5 Inverse Functions

Consider the function f has domain $A = \{1, 2, 3\}$ and range $B = \{3, 4, 5\}$ which is represented by a venn diagram as shown in **(i)**.




**(i)**




**(ii)**

A new function having domain B and range A is formed from the function f by reversing the arrows as shown in **(ii)**. This new function is called the inverse of f and is denoted by $f^{-1}$ (read as f inverse).

From the above diagrams, we observe and notice that

$$f(1) = 3, \qquad f^{-1}(3) = 1$$

$$f(2) = 4, \qquad f^{-1}(4) = 2$$

$$f(3) = 5, \qquad f^{-1}(5) = 3$$

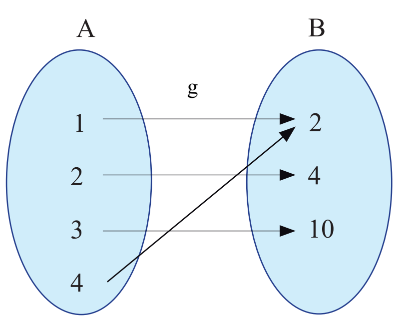


This establishes the following relation.

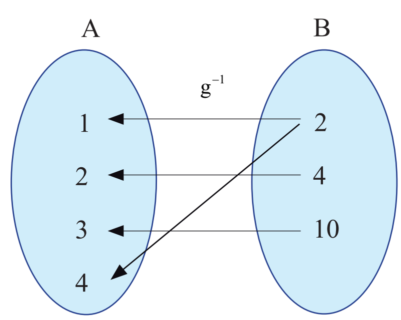
> The domain of the function f has become the range of the function $f^{-1}$ and the range of the function f has become the domain of the function $f^{-1}$.

Not all functions have an inverse. Let us illustrate this with an example.

Consider the following function g with domain $A = \{1, 2, 3, 4\}$ and range $B = \{2, 4, 10\}$ which is represented by a venn diagram as shown in **(iii)**.




**(iii)**




**(iv)**

The inverse function of g does not exist. This is because for any function to exist, each value of $x$ must give a unique value of $y$. However, in this case, 2 maps $g^{-1}$ to give 1 and 4. This is shown in **(iv)**.

From the discussion, we can conclude the following.

Suppose $f : A \mapsto B$ is a function from domain A to range B, then the inverse function of f exists only if f is one-one function.

It is defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y \qquad \text{for any } y \text{ in B.}$$

and also

Domain of f = Range of $f^{-1}$ and Range of f = Domain of $f^{-1}$.

**Footnote**

Consider $f^{-1}(y) = x$, if we want to call the independent variable $x$,

we then interchange $x$ and $y$ and have the following: $y = f^{-1}(x)$ for any $x$ in B.

The following should be noted when using the inverse function.

**1.** Do not regard $-1$ in $f^{-1}$ as to the power $-1$. Thus, $f^{-1}(x)$ does not mean $\dfrac{1}{f(x)}$.

**2.** The reciprocal $\dfrac{1}{f(x)}$ could, however, be written as $[f(x)]^{-1}$.



## Finding the inverse of a Function

We can find the inverse of a function using reversing mapping method. In this method, we know that if f maps $x$ onto $y$, then $f^{-1}$ maps $y$ onto $x$. Let us work with an example.

### Example 8 finding the inverse function

It is given that $f(x) = x+1$, where $x \in \mathbb{R}, x \geq 0$. Find the inverse function f and states its domain.

**Solution**

Suppose f maps $x$ onto $y$, then $\quad y = x+1$

Make $x$ in terms of $y$ gives

$$x = y-1$$

Now $f^{-1}$ maps $y$ onto $x$, therefore

$$f^{-1}(y) = y-1$$

So $\quad f^{-1}(x) = x-1 \quad \triangleleft$ replace $y$ by $x$

$\therefore$ the inverse of the function f is

$$\underline{\underline{f^{-1}(x) = x-1, \text{ where } x \in \mathbb{R}.}}$$ $\triangleleft$ notice that the domain of $f^{-1}$ which is the range of f

**FOLLOW UP 8**

The function is defined by $\mathrm{f}: x \mapsto 2x+1,\ x \in \mathbb{R}$. Find $\mathrm{f}^{-1}(x)$.

## Example 9 finding the inverse function

The function f is defined by $\mathrm{f}: x \mapsto x^2 - 2x + 3, x \in \mathbb{R}$. Find an expression for $\mathrm{f}^{-1}(x)$ for which

**(a)** $x \ge 1$ **(b)** $x \le 1$

### Solution

Let $y = x^2 - 2x + 3$

$= (x-1)^2 + 2$ ◁ completing the square

$(x-1)^2 = y - 2$ ◁ make $x$ in terms of $y$

$x^2 = 1 \pm \sqrt{y-2}$

**(a)** For $x \ge 1$,

$x = 1 + \sqrt{(y-2)}$

$\therefore$ the inverse function of f is

$\mathrm{f}^{-1}: x \mapsto 1 + \sqrt{x-2},\ x \ge 2$

**(b)** For $x \le 1$, $x = 1 - \sqrt{(y-2)}$

$\therefore$ the inverse function of f is

$\mathrm{f}^{-1}: x \mapsto 1 - \sqrt{x-2},\ x \ge 2$

**Big Idea**

Notice in **Example 9**, when a quadratic function is given, we use the method of completing the square to express $x$ in terms of $y$.



**FOLLOW UP 9**

The function f is defined by $\mathrm{f}: x \mapsto x^2 - 4,\ x \in \mathbb{R}$. find an expression of $\mathrm{f}^{-1}$ for which

**(a)** $x \ge 2$ **(b)** $x \le 2$

## Example 10 finding the inverse function

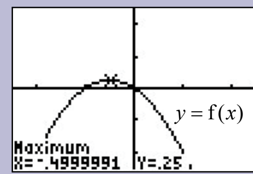
The function $\mathrm{f}: x \to -x^2 - x$, where $x \in \mathbb{R}, x \ge p$. Find

**(a)** the least value of $p$ so that the inverse function $\mathrm{f}^{-1}$ exists,

**(b)** the inverse function $\mathrm{f}^{-1}$ and states its domain.

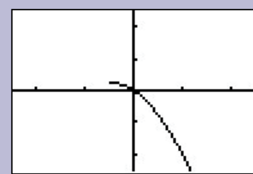
## Solution

**(a)** The graph of the function $f: x \to -x^2 - x$ for $x \in \mathbb{R}$ is shown.




When the function is at maximum, the value of $y$ is $\frac{1}{4}$.

Notice that when $y < \frac{1}{4}$, there are two values of $x$ for each value of $y$, therefore the graph does not represent a one-one function, thus the inverse function of f does not exist.



Therefore, we need to restrict the domain of function f such that it becomes a one-one function. To do so, we need to remove part of the curve to the left of its axis of symmetry, as shown.

The remaining domain of the function f is $x \geq -\frac{1}{2}$ and its range of f is $y \leq \frac{1}{4}$ (which will be the domain of $f^{-1}$).

$\therefore$ the least value of $p$ is $-\frac{1}{2}$.

**(b)** Let

$$y = -x^2 - x$$

$$= \frac{1}{4} - \left(x + \frac{1}{2}\right)^2 \quad \triangleleft \text{ completing the square}$$

$$x = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - y}$$

$\because x \geq -\frac{1}{2}$,

$$\therefore x = -\frac{1}{2} + \sqrt{\frac{1}{4} - y}$$

So the inverse function is

$$f^{-1}: x \mapsto -\frac{1}{2} + \sqrt{\frac{1}{4} - x}, \quad x \leq \frac{1}{4}.$$

**Big Idea**

Steps to find the inverse function:

**1**: Let $y = f(x)$

**2**: Find $x$ in terms of $y$.

**3**: Rewrite $x$ as $f^{-1}$ and replace all $y$ in terms of $x$.

**4**: State the domain of $f^{-1}$, which is the range of f.

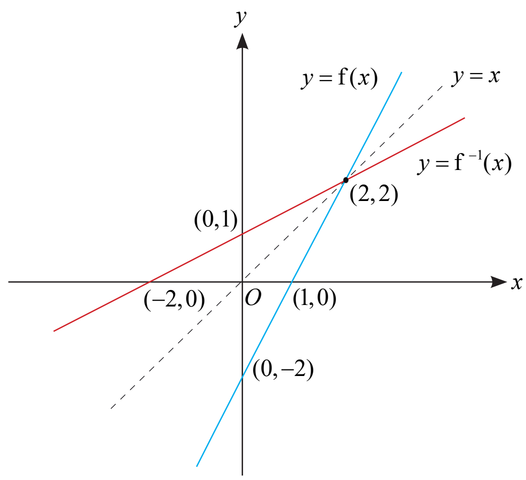
**Reflection**

In **Example 10**, the equation is written as $x = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - y}$.

Why do we reject $x = -\frac{1}{2} - \sqrt{\frac{1}{4} - y}$, when $x \geq -\frac{1}{2}$.

## Graphing Inverse Functions

Consider the function $f : x \to 2x - 2$, where $x \in \mathbb{R}$ and its inverse is $f^{-1} : x \to \frac{1}{2}(x+2)$, where $x \in \mathbb{R}$ and its graphs drawn on the same coordinates plane as shown.




We can see that the graphs of f (blue colour) and its inverse (red colour) are symmetrical about the line $y = x$. The point $(1, 0)$ lies on the graph f becomes $(0, 1)$ on the graph $f^{-1}$ after the reflection $y = x$.

The graphs of f and $f^{-1}$ interesect at the line $y = x$. In this example, the point of intersection of f and $f^{-1}$ is at (2, 2).

From the above discussion, we can establish the following relationship between the function f and $f^{-1}$.



If f is a one-one function, f has an inverse function $f^{-1}$, the graphs of f and $f^{-1}$ are *reflections* of each other in the line $y = x$.

The following table shows the geometrical effects between the graphs of the functions f and $f^{-1}$.

| The graph of f($x$) | The graph of $f^{-1}(x)$ |
|---|---|
| The point $P(x, y)$ lies on the graph of f. | The corresponding point becomes $(y, x)$ on the graph of $f^{-1}$. |
| For any vertical asymptote of the graph of f. | It becomes a horizontal asymptote of the graph $f^{-1}$. |
| The domain of the function f is D. | It becomes the range of the inverse function. |
| The range of the function of f is R. | It becomes the domain of the inverse funcion. |
| The two graphs intersect at $y = x$. | |

**FOLLOW UP 10**

The function $f : x \to e^{-x^2}$, where $x \in \mathbb{R}$.

**(a)** Sketch the graph of f. Hence state the range of f.

**(b)** Give a reason why f does not have an inverse.

The function f has an inverse if its domain is restricted to $x \geq k$.

**(c)** Find the least value of $k$ and define $f^{-1}$ in a similar form.

**(d)** Solve for $f(x) = f^{-1}(x)$, corresponding to the restricted domain in **(c)**.

## Example 11 using a graphic calculator to graph a function and its inverse

The function is defined by $g : x \mapsto 3-(x-1)^2$, for $x \leq 1$.

**(a)** Sketch the graphs of $y = g(x)$ and $y = g^{-1}(x)$ on a single diagram.

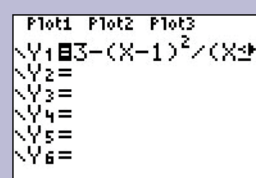
**(b)** Describe the relationship between the graphs.

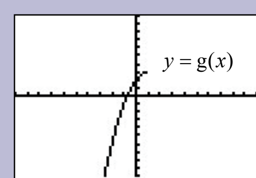
**(c)** By considering the relationship between the the graph of g and $g^{-1}$, find the exact value of $x$ which satisfies the equation $g(x) = g^{-1}(x)$ for $x \leq 1$.

### Solution

**(a)** Press [Y=] and enter the expression as shown.

Press [GRAPH] to obtain the graph of $y = g(x)$.

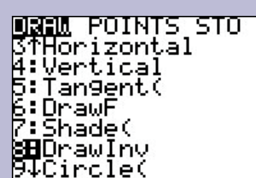


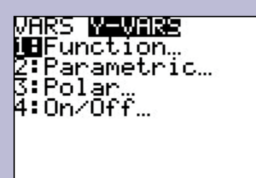




To call up inequalitiy symbol. Press [2nd] [MATH].



The following are the steps to graph the inverse function $g^{-1}(x)$ using a graphic calculator.




***Step1*** : [2nd] [PRGM] under sub-menu **'DRAW'** press [8].

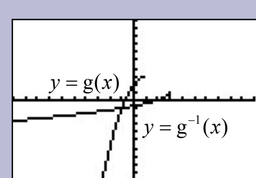
***Step2*** : Press [VARS] under submenu, select **Y - VARS** press [1].


***Step3*** : Press [1] follow by [ENTER].

**Note :** $Y_1$ refers to $g : x \mapsto 3-(x-1)^2$.

***Step 4*** : Press [ENTER] again.

The graph of $y = g^{-1}(x)$ is obtained.




**(b)** The graphs are reflections of each other in the line $y = x$.

**(c)** $\because$ g and $g^{-1}$ intersects at the line $y = x$, we equate

$$3-(x-1)^2 = x$$
$$x^2 - x - 2 = 0$$
$$\therefore \quad x = \underline{\underline{-1}} \quad \text{or} \quad x = 2 \quad (\text{Rejected} \because x \le 1)$$

**Reflection**

Notice in **Example 11(c)**, it is easier to find the point of intersection between f and its inverse using $f(x) = x$ or $f^{-1}(x) = x$ instead of $f(x) = f^{-1}(x)$.

**FOLLOW UP 11**

The function is defined by $f : x \mapsto 6 - e^{(x-1)^2}$, for $x \le 1$.

**(a)** Show that the function $f^{-1}$ exists and define $f^{-1}$ in similar form.

**(b)** Sketch the graphs of $y = g(x)$ and $y = g^{-1}(x)$ on a single diagram, stating the relationship between the graphs.

## 1.6 Composite Functions

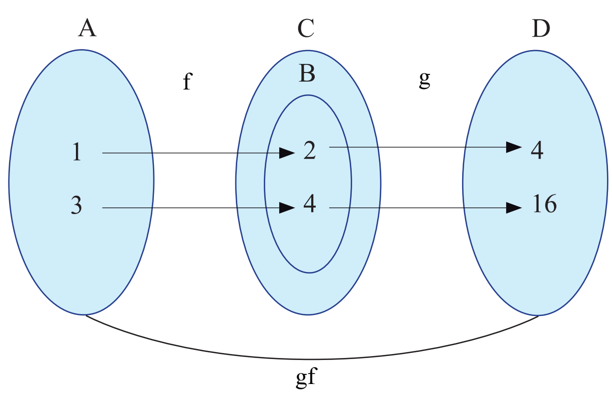


Suppose we have two functions f and g,
if the first function is f and the result of f is mapped onto g, the resultant (composite) function is called gf.

Consider the following functions

$$f(x) = x+1, \quad \text{has domain } A = \{1,3\} \text{ and range } B = \{2,4\}$$
$$g(x) = x^2, \quad \text{has domain } C = \{1,2,4,5\} \text{ and range } D = \{1,4,16,25\}$$

The venn diagram below shows how function gf is formed. function f first maps the elements in set A to set B where it is subset to those in set C. Another function g maps the elements in set B to those in set D.




**Big Idea**

When we form a composite function we can think of composition in terms of manufacturing a product. For example, fiber is first made of cloth, then the cloth is made into garment.

As shown:

$$1 \overset{f}{\mapsto} \underset{f(2)}{2} \overset{g}{\mapsto} \underset{gf(2)}{4}$$

$$3 \overset{f}{\mapsto} \underset{f(4)}{4} \overset{g}{\mapsto} \underset{gf(4)}{16}$$

$$x \overset{f}{\mapsto} \underset{f(x)}{x+1} \overset{g}{\mapsto} \underset{gf(x)}{(x+1)^2}$$

We interpret the above mapping by starting with a number in the domain f and find its image of $f(x)$.
If this number $f(x)$ is in the domain of g, then we calculate the value $gf(x)$.

For the element of $x$, the result is $(x+1)^2$, the composite function gf is formed.

i.e. $\quad gf : x \rightarrow (x+1)^2$

Strictly, a function is not completely defined unless we state the domain of the function. Since the elements of the function f map directly to the element of g, we say that Domain of f $=$ Domain of gf.
In this example, the domain of the function gf is [1,3].

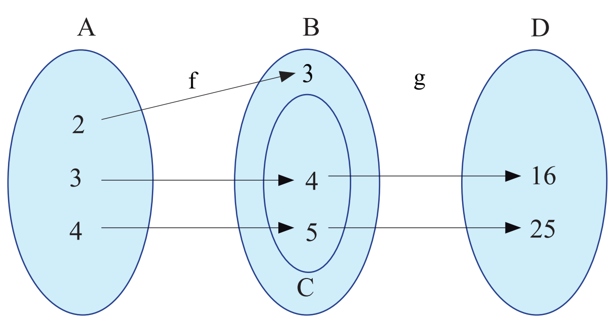
Using the venn diagram shown on the previous page, we can also obtain the range of the function gf.
$\therefore$ the range for the function gf is [4,16].



Not all composite functions can be formed, as seen in the functions below.

$f : (x) \mapsto x+1$, with domain $A = \{2,3,4\}$ and range $B = \{3,4,5\}$

and $g : (x) \mapsto x^2$, with domain $C = \{4,5\}$ and range $D = \{16,25\}$




From the diagrams, we see that 2 maps f to give 3, which is not in the domain C
Thus, the composite function gf does not exist.

Summing up the above disscusion, we conclude as follows.

To form composite function gf, the range of f must be included (subset) in the domain of g, i.e. $R_f \subseteq D_g$

The function gf is then defined as

$$gf : x \mapsto g(f(x)),\ x \in \text{domain of f.}$$

Algebraically, the composite function gf is obtained by substituting function f into g to obtain composite function gf since the first function is f and the second function is g.

For example, if functions f and g are defined by $f : x \mapsto x+1,\ x \in \mathbb{R}$ and $x > 0$ and $g : x \mapsto x^2, x \in \mathbb{R}$, then the composite function gf can be expressed as follows.

$$\begin{aligned} gf(x) &= g[f(x)] \quad \triangleleft \text{ replace } f(x) \text{ by } x+1. \\ &= g(x+1) \\ &= (x+1)^2 \end{aligned}$$

## Example 12 finding the composite functions



Functions f and g are defined by $f : x \mapsto 2x^3,\ x \in \mathbb{R}$ and $g : x \mapsto 3x+1,\ x \in \mathbb{R}$. Find the following expression for

**(a)** fg **(b)** gf **(c)** ff

### Solution

**(a)**

$$\begin{aligned} fg(x) &= f[g(x)] \\ &= f(3x+1) \\ &= 2(3x+1)^3 \end{aligned}$$

$\therefore\ \underline{\underline{fg : x \mapsto 2(3x+1)^3,\ x \in \mathbb{R}.}}$

**(b)**

$$\begin{aligned} gf(x) &= g[f(x)] \\ &= g[2x^3] \\ &= 3(2x^3)+1 \end{aligned}$$

$\therefore\ \underline{\underline{gf : x \mapsto 6x^3+1,\ x \in \mathbb{R}.}}$

***Commentary*** :

We can see that the composite functions fg and gf are two different functions. Thus, fg is not equal to gf.

**Reflection**

Suppose the graphs f and g are lines. Must it be true that composite function fg is a line? Justify your answer.

**Information**

If $a$ is a real number and $\mathbb{R}$ represents a set of real numbers, then we say $a$ is an element of $\mathbb{R}$. In notation, we write $a \in \mathbb{R}$.

**(c)** Let $y = 2x^3$ ◁ express $x$ in terms of $y$

$$\frac{y}{2} = x^3$$

$$x = \sqrt[3]{\frac{y}{2}}$$

So the inverse function of f is $f^{-1} : x \mapsto \sqrt[3]{\frac{x}{2}},\ x \in \mathbb{R}$

$$ff^{-1} = f[f^{-1}(x)]$$
$$= f\left(\sqrt[3]{\frac{x}{2}}\right)$$

$\therefore$ $ff^{-1} : x \mapsto x,\ x \in \mathbb{R}$

***Commentary*** :

Functions such as $f^{-1}f$ and $ff^{-1}$ are called identity functions because the domain and the range of these functions are the same.

For example, $ff^{-1}(x) = x$, where $x \in$ domain of f.

**Reflection**

If the composite function fg exists, is it true to say that the composite function gf exists too?

**FOLLOW UP 12**

Functions f and g are defined by $f : x \mapsto 2x + 3,\ x \in \mathbb{R}$ and $g : x \mapsto x^2 - 1,\ x \in \mathbb{R}$. Find

**(a)** $fg(x)$ **(b)** $gf(x)$ **(c)** $ff^{-1}(x)$



## Example 13 showing that composite function exists

Functions f and g are defined by

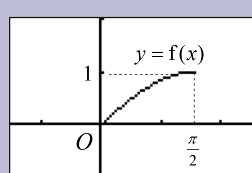
$$f : x \mapsto \sin x,\ x \in \left(0, \frac{\pi}{2}\right]$$

$$g : x \mapsto x^2,\ x \in \mathbb{R}$$

**(a)** Sketch the graph of the function f. State the range of f.
**(b)** Show that composite function gf exists and define gf in similar form.
**(c)** State the range of gf.

### Solution

**(a)** From the diagram on the right, we can see that the least value of the graph is 0 and the greatest value of the graph is 1.




$\therefore$ the range of the f is $(0, 1]$.

**(b)** ***Commentary*** :

To show that a composite function, say gf to exist, we need to show $R_f \subseteq D_g$.

Begin by stating the range of f and the domain of g and we have the range of f as

$$R_f = (0, 1]$$

and the domain of g as

$$D_g = \mathbb{R} \quad \triangleleft \text{ given}$$

From the above, we can see that the range of f is a subset to the domain of g, i.e. $R_f \subseteq D_g$.

$\therefore$ the function gf exists.

> **Information**
>
> If all the elements of set $A$ are elements of the set $B$, we say that $A$ is a subset of $B$ and write $A \subseteq B$.

**(c)** There are *two methods* to find the range of a composite function. They are the **Graphical** method and the **Mapping** method.

**Graphical Method**

Begin by finding the composite function of gf which gives

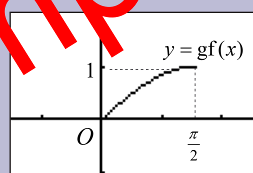
$$\begin{aligned} gf(x) &= g[f(x)] \\ &= g[\sin x] \\ &= (\sin x)^2 \end{aligned}$$

$\therefore$ the function gf is $gf : x \mapsto (\sin x)^2,\ x \in \left(0, \frac{\pi}{2}\right]$. $\quad \triangleleft$ domain of gf = the domain of f

> **Recall**
>
> Recall that $\sin^2 x = (\sin x)^2$,
> However, $\sin^2 x$ is **not** equal to $\sin x^2$.

Using a graphic calculator to graph function gf.
From the diagram, we can see the least value of gf is 0 and the greatest value of gf is 1.

$\therefore$ the range of gf is (0, 1].





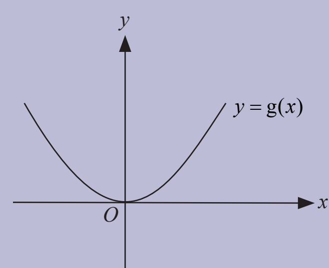

**Mapping Method**

The range of the composite function, gf can be determined by using the mapping method.
Begin by finding the range of f.

$$\underbrace{\left(0, \frac{\pi}{2}\right]}_{D_f} \xrightarrow{f} \underbrace{(0, 1]}_{R_f = \text{Restricted } D_g}$$

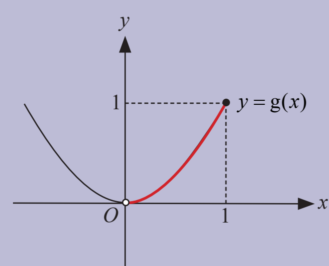
> **Attention**
>
> Notice in **Example 13**, the domain of gf is equal to the domain of f. However, the range of gf may not necessarily be equal to the range of f.

Sketch the graph of g, as shown.




Next, restrict the domain g by taking the range of f as the new domain to obtain the range of gf (red part)

$$\underbrace{\left(0, \frac{\pi}{2}\right]}_{D_f} \xrightarrow{f} \underbrace{(0, 1]}_{R_f = \text{Restricted } D_g} \xrightarrow{g} \underbrace{(0, 1]}_{R_{gf}}$$




$\therefore$ the range of gf is (0, 1].

**FOLLOW UP 13**

Functions f and g are defined by

$$\mathrm{f}: x \mapsto \frac{1}{x^2+1},\ x \in \mathbb{R},\ x \le 0$$

$$\mathrm{g}: x \mapsto \frac{1}{x},\ x \in \mathbb{R},\ x > 0.$$

**(a)** Sketch the graph of the function f.

**(b)** Show that composite function gf exists.

**(c)** Find $\mathrm{gf}(x)$ and state the domain and range of gf.

## Example 14 showing the composite function exists

The functions f and g are defined by

$$\mathrm{f}: x \mapsto \frac{x+2}{x+1},\ x \in \mathbb{R} \text{ and } x \ge 0$$

$$\mathrm{g}: x \mapsto x^2 - 2x + 2,\quad x \in \mathbb{R} \text{ and } 0 \le x \le 1$$

**(a)** Show that the composite function fg exists.

**(b)** Determine its rule for fg and its domain.

**(c)** Find the range of fg in exact form.

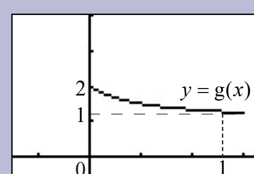


### Solution

***Commentary*** : To show the composite function fg exists, we need to show that the range of g must be included (subset) to the domain of f, i.e. $R_g \subseteq D_f$.

**(a)** Begin by using a graphic calculator to determine the range of g, which gives

$$R_g = [1, 2]$$




From the function f, we have the domain of f which is

$$D_f = [0, \infty)$$

$\because$ the range of g is a subset to the domain of f, i.e. $R_g \subseteq D_f$,

$\therefore$ the composite function fg exists.

**(b)**

$$\mathrm{fg}(x) = \mathrm{g}(x^2 - 2x + 2)$$

$$= \frac{(x^2 - 2x + 2) + 2}{(x^2 - 2x + 2) + 1}$$

$\therefore$ the composition function fg is

$$\mathrm{fg}: x \mapsto \frac{x^2 - 2x + 4}{x^2 - 2x + 3},\ x \in \mathbb{R},\ 0 \le x \le 1.$$

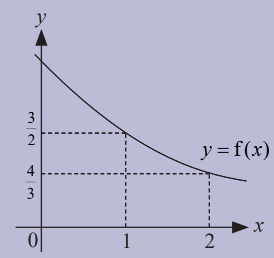
**Information**

The domain of gf = the domain of f.
Similarly,
the domain of fg = the domain of g.

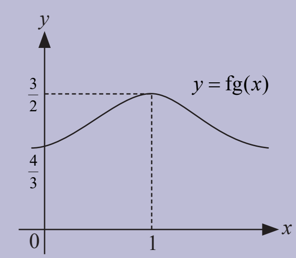
**(c)** Using the mapping method, we have the range of gf as

$$\underbrace{[0,1]}_{D_g} \xrightarrow{g} \underbrace{[1,2]}_{R_g = \text{Restricted } D_f} \xrightarrow{f} \underbrace{\left[\frac{4}{3}, \frac{3}{2}\right]}_{R_{fg}}$$

$\therefore$ the range of fg is $\left[\frac{4}{3}, \frac{3}{2}\right]$.




Alternatively, we can graph the function gf to obtain the range of gf as shown on the right.




**FOLLOW UP 14**

A function f is defined by

$$f : x \mapsto \frac{1}{2}e^{1-x^2}, \; x \in \mathbb{R} \text{ and } x \le -1$$

**(a)** Justify, with a reason whether $f^{-1}$ exists.

**(b)** If the domain is restricted to $(-\infty, b]$ such that $b$ is the largest value for which the inverse function $f^{-1}$ exists, states the value of $b$ and find $f^{-1}(x)$, stating its domain.

Another function g is defined by

$$g : x \mapsto \sqrt{x}, \;\; x \in \mathbb{R} \text{ and } \; 0 \le x \le 1.$$

**(c)** Show that gf exists.

**(d)** Find $gf(x)$, stating its domain.

**(e)** Find the exact range of gf.



## Example 15 showing composite function does not exist

The function f is defined as

$$f : x \mapsto \frac{x}{x-1}, \; x \in \mathbb{R}, \; x > 1$$

**(a)** Find $f^{-1}(x)$, stating the domain of $f^{-1}$.

**(b)** Find $f^2(x)$ and $f^3(x)$. Hence evaluate $f^5(4)$.

Another function g is defined as

$$g : x \mapsto e^x, \; x \in \mathbb{R}$$

**(c)** Determine whether the composite function fg exists, justifying your answer.

## Solution

**(a)** Let $y = \mathrm{f}(x)$

$$y = \frac{x}{x-1}$$

$$y(x-1) = x \quad \triangleleft \text{ express } x \text{ in terms of } y$$

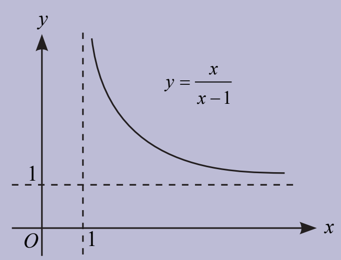
$$xy - y = x$$

$$xy - x = y$$

$$x(y-1) = y$$

$$x = \frac{y}{y-1}$$

$$\therefore\ \mathrm{f}^{-1} : x \mapsto \frac{x}{x-1},\ x \in \mathbb{R},\ x > 1$$




***Commentary*** :

If $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$, then f is called a self-inverse function. In **(a)**, we note that $\mathrm{f}(x) = \frac{x}{x-1}$ for $x > 1$ and then $\mathrm{f}^{-1}(x) = \frac{x}{x-1}$ for $x > 1$.

Hence, $\mathrm{f}(x) = \frac{x}{x-1}$, for $x > 1$ is an example of a self-inverse function.

**(b)** Since $\mathrm{f}(x)$ is a self-inverse function, we have

$$\mathrm{f}(x) = \mathrm{f}^{-1}(x)$$

so $$\mathrm{ff}(x) = \mathrm{ff}^{-1}(x) \quad \triangleleft \text{ introduce f on both sides}$$

$$\mathrm{ff}(x) = x$$

$$\therefore\ \mathrm{f}^2(x) = x$$

Consider $$\mathrm{f}^2(x) = x$$

$$\mathrm{ff}^2(x) = \mathrm{f}(x) \quad \triangleleft \text{ introduce f on both sides}$$

$$\mathrm{f}^3(x) = \frac{x}{x-1}$$

$$\therefore\ \mathrm{f}^3(x) = \frac{x}{x-1}$$



From the above observation, we note the following.

When $n$ is odd, $\mathrm{f}^n(x) = \frac{x}{x-1}$ and when $n$ is even, $\mathrm{f}^n(x) = x$.

So, $\mathrm{f}^5(x) = \frac{x}{x-1}$

$$\therefore \quad \mathrm{f}^5(4) = \frac{4}{4-1} \quad \triangleleft \text{ substitute } x = 4$$

$$= \frac{4}{3}$$

**Reflection**

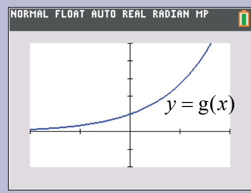
Consider function

$\mathrm{f}(x) = \frac{ax+b}{cx+d},\ x \neq -\frac{d}{c}.$

State a condition such that f will not be a self - inverse.

**(c)** Begin by using a graphic calculator to determine the range of g that gives

$$R_g = (0, \infty)$$




From the function f, the domain of f is given as $D_f = [1, \infty)$.

From the above, we can see that the range of g is not a subset to the domain of g, i.e. $R_g \not\subset D_f$.

$\therefore$ the composite function fg does not exist.

**FOLLOW UP 15**

A function f is said to self-inverse if $f(x) = f^{-1}(x)$ for all domain of $x$.
The functions f and g are defined by

$$f : x \mapsto \frac{7-3x}{3-x}, \qquad x \in \mathbb{R},\ x \neq 3$$

$$g : x \mapsto \ln(x+1), \qquad x \in \mathbb{R},\ -1 < x < 2.$$

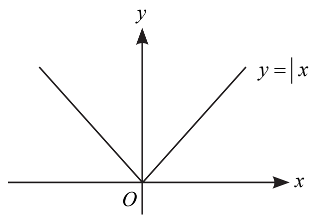
**(a)** Show that f is self-inverse. Hence evaluate $f^{2003}(5)$.
**(b)** Find an expression for $g^{-1}(x)$.
**(c)** Explain why composite function $gf^{-1}$ does not exist.
**(d)** Show that $fg^{-1}$ exists. Find the exact range $fg^{-1}$.



## Piecewise Functions

A piecewise function is a function which makes up of 2 or more sub-functions, where each sub-function has a different interval in the domain. An example of a piecewise function is the modulus function.
It is defined as

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$




## Example 16 graphing piecewise function

The function f is defined as

$$f(x) = \begin{cases} x(x-2) & \text{for } 0 \leq x \leq 1, \\ 2-x & \text{for } 1 \leq x < 2. \end{cases}$$

**(a)** Sketch the graph of $f(x)$ for $0 \leq x < 2$.
**(b)** Find the range of $f(x)$.

The function g is defined as

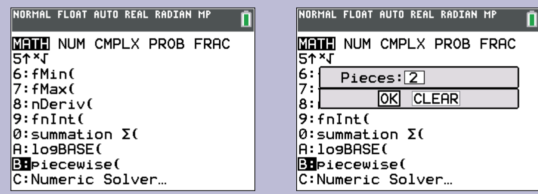
$$g(x) = e^x,\ x \in \mathbb{R},\ x \geq 0.$$

**(c)** Show that gf exists.
**(d)** Give a definition (including the domain) of the composite function gf and its range.

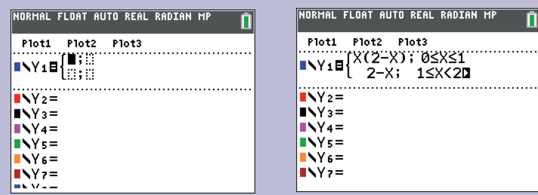
## Solution

**(a)** The steps to graph a piecewise function using a graphic calculator are as follows.

***Step1*** : Press **Y=** **MATH** under sub-menu **'MATH'** scroll piecewise. Press **ENTER**.

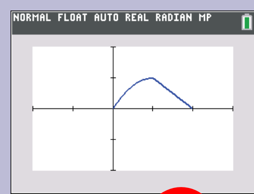



***Step2*** : Select **2** (since it is a 2 sub-functions) and press **'OK'**.

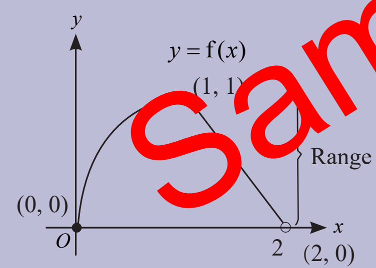



***Step 3*** : Input the functions and its domain as shown.

Press **GRAPH**.




The graph of the piecewise function f is shown below.






***Commentary*** :

To graph a piecewise function, we need to ensure all the details such as stating the end-points on the graph.

**(b)** From the graph, the range of $f(x)$ is $\underline{\underline{0 \leq f(x) \leq 1}}$.

**(c)** From **(b)**, $R_f = [0,\ 1]$ and $D_g = [0,\ \infty)$ which is given.

Since $R_f \subseteq D_g$,

$\therefore$ $\underline{\underline{\text{gf exists}}}$.

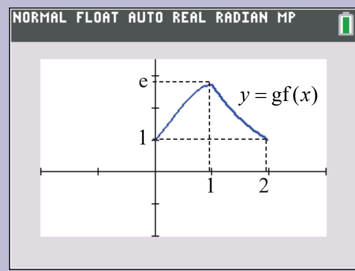
**(d)** For $0 \le x \le 1$,

$$\begin{aligned} \mathrm{gf}(x) &= \mathrm{g}[\mathrm{f}(x)] \\ &= \mathrm{g}[x(x-2)] \\ &= \mathrm{e}^{x(x-2)} \end{aligned}$$

For $1 \le x < 2$,

$$\begin{aligned} \mathrm{gf}(x) &= \mathrm{g}[\mathrm{f}(x)] \\ &= \mathrm{g}[2-x] \\ &= \mathrm{e}^{2-x} \end{aligned}$$

$$\therefore\ \mathrm{gf}(x) = \begin{cases} \mathrm{e}^{x(x-2)} & \text{for } 0 \le x \le 1, \\ \mathrm{e}^{2-x} & \text{for } 1 \le x < 2. \end{cases} \quad \triangleleft\ \mathrm{D_{gf}} = \mathrm{D_f}$$




Using graphic calculator to graph function $\mathrm{gf}(x)$.

$\therefore$ range of gf $= [1,\ \mathrm{e}]$.

**FOLLOW UP 16**

The function f is defined by

$$\mathrm{f} : x \to \begin{cases} 1 - x^3 & \text{for } -2 < x < 0, \\ \mathrm{e}^{-2x} & \text{for } x \ge 0. \end{cases}$$

**(a)** Sketch the graph of f and state the range of f.

**(b)** Find $\mathrm{f}^{-1}$ in a similar form.

The function g is defined by

$$\mathrm{g} : x \to x(x-2),\ x \in \mathbb{R}$$

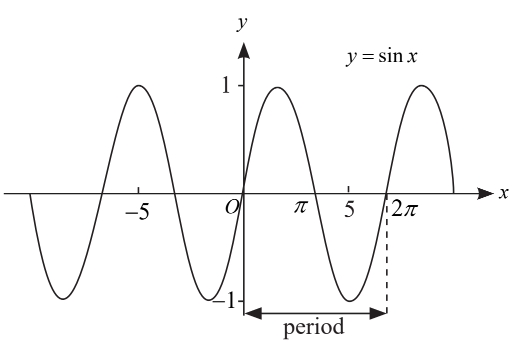
**(c)** Show that the function fg exists.

**(d)** Given that $\alpha > 2$, find an expression for $\mathrm{fg}(\alpha)$. Hence, find the value of $\alpha$ for which $\mathrm{fg}(\alpha) = \frac{1}{4}$.

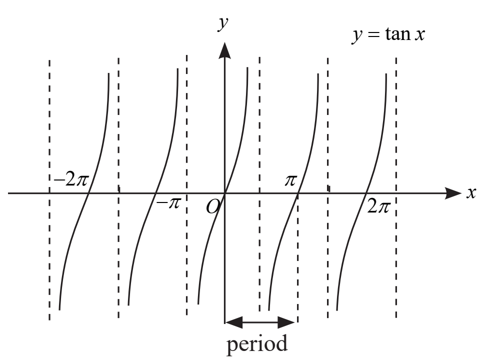


## Periodic Functions

A function is said to be periodic when its graph consists of a certain pattern which repeats at a regular interval. The graphs of some common periodic functions are shown below.




sin $x$ is periodic and its period is $2\pi$.




tan $x$ is periodic and its period is $\pi$.

For a period function with a period $a$,

it can written as $\quad \mathrm{f}(x) = \mathrm{f}(x+a) = \mathrm{f}(x+2a) = \mathrm{f}(x+3a) = \ldots$

similarly, $\quad \mathrm{f}(x) = \mathrm{f}(x-a) = \mathrm{f}(x-2a) = \mathrm{f}(x-3a) = \ldots$

So, a periodic function is defined by the condition

$\mathrm{f}(x) = \mathrm{f}(x \pm ka)$, where $k$ is a non-zero constant and $a$ is the period of the function.

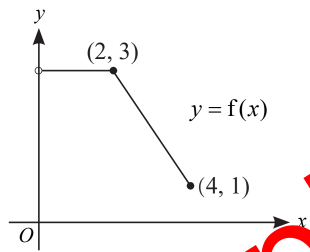
The pattern in the graph of a period function may be made up of two or more different definitions.

For example, a function is defined by

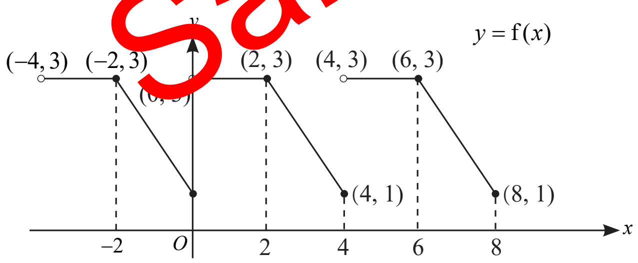
$$\mathrm{f}(x) = \begin{cases} 3, & \text{for } 0 \le x \le 2 \\ 5 - x & \text{for } 2 \le x \le 4 \end{cases}$$

and that $\mathrm{f}(x) = \mathrm{f}(x+4)$ for all real values of $x$.

The graph below represents the function $y = \mathrm{f}(x)$, for $0 \le x \le 4$.




Given that $y = \mathrm{f}(x) = \mathrm{f}(x+4)$ for all real values of $x$, it tells us that the function f is a periodic with a period of 4. Therefore, we repeat the above graph at the interval of 4 units each time. This is shown in the graph below.






## Example 17 graphing periodic function

It is given that

$$\mathrm{f}(x) = \begin{cases} x^2 + 1 & \text{for } 0 \le x \le 3 \\ 4 - x & \text{for } 3 \le x \le 4 \end{cases}$$

and that $\mathrm{f}(x) = \mathrm{f}(x+4)$ for all real values of $x$.

**(a)** State a reason why f is not a 1-1 function.

**(b)** Sketch the graph of $\mathrm{f}(x)$ for $-2 \le x \le 4$.

**(c)** Find $\mathrm{f}(13)$.

## Solution

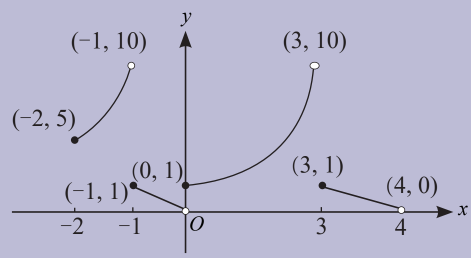
**(a)** It is given that $f(x) = f(x+4)$ for all real values of $x$.

Consider $x = 1$, $\quad f(1) = f(1+4)$

$\qquad\qquad\qquad\quad f(1) = f(5)$

For one-one function to exist, every value of $f(x)$ there is only one value of $x$. Since there are two values of $x$ give the same value of $f(x)$. Hence, f is not a 1-1 function.

**(b)**




**(c)** Given that $f(x) = f(x+4)$ for all real values of $x$,

it means that $f(13) = f(9) = f(5) = f(1)$

So, $f(1) = 1^2 + 1 = 2$

$\therefore \underline{\underline{f(13) = 2}}$



### FOLLOW UP 17

It is given that

$$f(x) = \begin{cases} x+1 & \text{for } 0 \le x < 2 \\ 7-x & \text{for } 2 \le x \le 4 \end{cases}$$

and that $f(x) = f(x+4)$ for all real values of $x$.

**(a)** Find $f(1)$, $f(8)$ and $f(29)$.

**(b)** Sketch the graph of $f(x)$ for $-3 \le x \le 8$.

**(c)** State a reason why f is not a 1-1 function.

# Chapter Summary

**1. Definition of Function**

Suppose X and Y are two sets of elements. A function $f : X \mapsto Y$ is a relation such that each element of set X maps exactly one definite element of set Y.

**(a) Vertical Line Test**

A function exists if a vertical line (within the domain of the function) cuts the graph of the function at **exactly** one point.

**(b) Domain of a Function**

The set of numbers $x$ for which a function $f(x)$ is defined is called the domain of the function.

**(c) Range of a Function**

The range of the function f refers to the sets of values of $y$ such that $y = f(x)$, can take from the numbers $x$, where $x \in$ domain of f.

**2. One - One Functions**

A function f defined for some domain D is one-one if **each value** of $y$ in the range of f there is only **one value** $x \in D$ such that $y = f(x)$.

**(a) 1 - 1 Function Test (Horizontal line test)**

One-One function exists when a **horizontal line** (within the range of the function) cuts the graph of function f at **exactly** one point.

**(b)** To show that a function is not one-one, state and provide one horizontal line $y = k$ (use a specific value of $k$) that intersects the graph more than once.

**3. Inverse Functions**

If f is a function, then $f^{-1}$ is called the inverse function of f.

**(a)** The inverse function f exists only if f is a one-one function.

**(b)** The domain of function f = the range of function $f^{-1}$. i.e $D_f = R_{f^{-1}}$.

**(c)** The range of function f = the domain of function f, i.e $R_f = D_{f^{-1}}$.

**(d) Relationship between the graphs of f and $f^{-1}$**

**(i)** They *reflect* in the line $y = x$.

**(ii)** They *intersect* at $y = x$.

**4. Composite Functions**

A function formed by composing two functions is called the composite function.

**(a)** The composite function fg exists if and only if $R_g \subseteq D_f$.

**(b)** If fg is a composite function, then $D_{fg} = D_g$ and $R_{fg} \subseteq R_f$.

**(c)** Suppose the function gf exists,then it is defined as $gf : x \mapsto g(f(x))$, $x \in$ domain of f.

## Exercise 1

### A Showing Functions exist

**1.** For each of the following graph, determine which are functions.

**(a)** $y = 2x^2 + x + 3$ **(b)** $y^2 + x^2 = 9$ **(c)** $y = \dfrac{2}{(x-3)^2}$ **(d)** $y^2 = 1 - x$

### B Domain of a Function

**2.** Find the largest possible domain for each of the following functions.

**(a)** $y = \sqrt{x}$ **(b)** $y = \sqrt{x(x-4)}$ **(c)** $y = \dfrac{1}{(x+1)(x-2)}$

### C Range of a Function

**3.** The functions f and g are defined by $\mathrm{f} : x \mapsto \dfrac{2x+1}{x+1},\ x > 0$ and $\mathrm{g} : x \mapsto -1 + \ln(x+2),\ x > 0$.

**(a)** Sketch, on separate diagrams, the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$.

**(b)** State the range of the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$.

**4.** Three functions f, g and h are given by $\mathrm{f} : x \mapsto \sqrt{16 - x^2},\ x \in \mathbb{R},\ -4 \le x \le 0,\ \mathrm{g} : x \mapsto \ln(2+x),\ x \in \mathbb{R},\ x > -2$ and $\mathrm{h} : x \mapsto 2 + e^{-x},\ x \in \mathbb{R}^+$.

**(a)** Sketch on separate diagrams, the graphs of f, g and h.

**(b)** State the range of all the three functions.

**5.** **(a)** A function f is defined by $\mathrm{f} : x \mapsto (x-a)(x-b),\ x \in \mathbb{R},\ x > b$, where $0 < a < b$. Sketch the graph of f and find its range.

**(b)** The function g is defined by $\mathrm{g} : x \mapsto \dfrac{2x+a}{x-b}$ for $x \in \mathbb{R},\ x > b,$ where $a, b \in \mathbb{R}^+$. Find the range of g.



### D One - one Functions and Inverse Functions

**6.** The function f is defined by

$$\mathrm{f} : x \mapsto -\ln(1+x),\ x \in \mathbb{R},\ -1 < x \le 2.$$

**(a)** Show that f is a one-one function.

**(b)** Find $\mathrm{f}^{-1}(x)$ and state the domain of $\mathrm{f}^{-1}$.

**(c)** It is given that $\mathrm{f}^{-1}(x) = 1,$ state the value of $x$.

**7.** Function h is defined as follows: $\mathrm{h} : x \to -x^2 + 3x + 14,\ x \in \mathbb{R}$.

**(a)** Show by means of a graphical argument that the function h is not one to one.

**(b)** The function h has an inverse if its domain is restricted to $x \le k$. Find the greatest value of $k$ for which the inverse function $\mathrm{h}^{-1}$ exists and define $\mathrm{h}^{-1}$, stating its domain.

**8.** The function f is defined by $\mathrm{f} : x \mapsto \left|(2x+1)(2x-9)\right|,\ x \in \mathbb{R}$.

**(a)** Given that $\mathrm{f}^{-1}$ exist when domain is restricted to $[a,\ b]$ where $a < 0$ and $b > 0,$ find the least value of $a$ and greatest value of $b$.

Using the restricted domain found in part **(a)**,

**(b)** find the domain of $\mathrm{f}^{-1}$ and the expression for $\mathrm{f}^{-1}(x)$.

**9.** The function f is defined as follows: $\mathrm{f}: x \mapsto \mathrm{e}^{-2x}, x > 0$.

**(a)** Show that f is one-one and define $\mathrm{f}^{-1}$. State the domain and the range of $\mathrm{f}^{-1}$.

**(b)** On the same axes, sketch the graphs of f, $\mathrm{f}^{-1}$ and $\mathrm{ff}^{-1}$ and state a relationship between the graphs of f and $\mathrm{f}^{-1}$.

**(c)** Explain why the $x$-coordinate of the point of intersection of the curves in part **(b)** satisfy the equation $\ln x - 2x = 0$.

**(d)** Hence, find the value of $x$ such that $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$, correct to 4 significant figures.

**10.** The function f is defined as

$$\mathrm{f}: x \mapsto \sqrt{x+1} - \frac{1}{2}, \quad x \in \mathbb{R}, x > -1.$$

**(a)** Sketch the graph of $y = \mathrm{f}(x)$. Your sketch should state the coordinates of any points of intersection with the axes and endpoints.

**(b)** Find $\mathrm{f}^{-1}(x)$, stating the domain of $\mathrm{f}^{-1}$. Hence the range of $\mathrm{f}^{-1}$.

**(c)** On the same diagram as in part **(a)**, sketch the graph of $y = \mathrm{f}^{-1}(x)$.

**(d)** Write down the equation of the line in which the graph of $y = \mathrm{f}(x)$ must be reflected in order to obtain the graph of $y = \mathrm{f}^{-1}(x)$, and hence find the exact solution of the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$.

**(e)** Using **(c)** and **(d)** to deduce the solution set of $\mathrm{f}(x) \geq \mathrm{f}^{-1}(x)$.

## E Composite Functions



**11.** The functions f and g are defined as follows: $\mathrm{f}: x \mapsto x^2, x \in \mathbb{R}$ and $\mathrm{g}: x \mapsto x^2 + 4,\ x \in \mathbb{R}$. Find

**(a)** the function $\mathrm{fg}(x)$,

**(b)** the function $\mathrm{gf}(x)$,

**(c)** the function of $\mathrm{f}^2(x)$.

**12.** A function f is given by $\mathrm{f}: x \mapsto x+1$. Find the function g in each of the following cases:

**(a)** $\mathrm{gf}: x \mapsto x^2$ **(b)** $\mathrm{fg}: x \mapsto x^2$

**13.** The functions f and g are defined by

$$\mathrm{f}: x \mapsto \frac{x+1}{x},\ x \in \mathbb{R},\ x > 0,$$

$$\mathrm{g}: x \mapsto -1 + \ln(x+2),\ x \in \mathbb{R},\ x > 0.$$

**(a)** The function h is such that $\mathrm{hg}(x) = \mathrm{e}^{-1}(x+2)$. Find an expression for $\mathrm{h}(x)$.

**(b)** Find $\mathrm{fh}(x)$ and state its domain.

**14.** Functions f and g are defined by

$$\mathrm{f}: x \mapsto x^2 - 4x + 3 \quad \text{for} \quad x \in \mathbb{R}, x < 2,$$

$$\mathrm{g}: x \mapsto \ln(x^2+1) \quad \text{for} \quad x \in \mathbb{R}.$$

**(a)** Only one of the composite functions fg and gf exists. Give a definition (including the domain) of the composite that exists, and explain why the other composite does not exist.

**(b)** Find the range of the composite function that exists.

**15.** Functions f and g are defined by

$$\mathrm{f}: x \mapsto 1+\frac{2}{x-1}, \quad x \in \mathbb{R}, x<1,$$
$$\mathrm{g}: x \mapsto \ln x, \qquad x \in \mathbb{R}, 0<x<1.$$

**(a)** Explain why the composite function gf does not exist.

**(b)** Find an expression for fg($x$). Hence or otherwise, find $(\mathrm{fg})^{-1}(0)$.

**(c)** Find an expression for h($x$) for each of the following cases:

**(i)** $\mathrm{gh}(x)=x$,

**(ii)** $\mathrm{gh}(x)=x^2+1$.

**16.** The functions f and g are defined by

$$\mathrm{f}: x \mapsto \sin x, \qquad x \in \mathbb{R}, -\pi \le x \le \pi$$
$$\mathrm{g}: x \mapsto 2x^2+x, \quad x \in \mathbb{R}, x \ge -1$$

**(a)** Show that the function gf exists. Express gf in a similar form and state its range.

**(b)** If $\mathrm{g}_1$ is a restriction of g, define the domain of $\mathrm{g}_1$ such that $\mathrm{g}_1^{-1}$ exists and $\mathrm{g}_1$ has the same range as g.

**17.** A function f is said to be a self-inverse function if $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$.

The function f is defined by

$$\mathrm{f}(x)=\frac{ax+b}{cx-2}, \quad x \in \mathbb{R}, x \ne \frac{2}{c},$$

where $a, b, c$ and $d$ are non-zero constants.

**(a)** By finding $\mathrm{f}^{-1}(x)$, show that $a=2$ for f to be a self-inverse function.

For the rest of the question, use $a=2$, $b=3$ and $c=5$.

**(b)** Find $\mathrm{f}^2(x)$.

**(c)** Evaluate $\mathrm{f}^{71}(4)$.

The function g is defined by

$$\mathrm{g}(x)=2x^2-3, \quad x \in \mathbb{R}.$$

**(d)** Explain why composite function fg does not exist.

**(e)** Find the exact solutions of $\mathrm{gf}(x)=5$.



**18.** A function h is said to self-inverse if $\mathrm{h}(x)=\mathrm{h}^{-1}(x)$ for all $x$ in the domain of h.

Functions f and g are defined by

$$\mathrm{f}: x \mapsto \frac{5x-3}{x-5}, \quad x \in \mathbb{R}, x \ne a, \text{ where } a \text{ is a constant},$$
$$\mathrm{g}: x \mapsto \ln x, \qquad x \in \mathbb{R}, x \ge e^{10}.$$

**(a)** State the value of $a$ and explain why this value has to be excluded from the domain of f.

**(b)** Show that f is self-inverse.

**(c)** Find the exact values of $b$ such that $\mathrm{f}^4(b)-2=\mathrm{f}^{-1}(b)$.

**(d)** Find the exact range of fg.

**19.** The functions f and g are defined by

$$\mathrm{f}: x \mapsto \ln(x+h+1), \quad x \ge -h, \;\; h>0$$
$$\mathrm{g}: x \mapsto x^2+2x-1, \quad x \in \mathbb{R}.$$

**(a)** Find the set of values of $h$ such that the composite functions fg exists.

**(b)** Find the value of $h$ such that the range of fg is given by $[\ln 3, \infty)$.

**20.** The functions f and g are defined by

$$\mathrm{f} : x \mapsto x^2 + 3x + 1, \quad x \in \mathbb{R}, x \le -\frac{3}{2},$$

$$\mathrm{g} : x \mapsto 3 + \mathrm{e}^{-x}, \quad x \in \mathbb{R}.$$

**(a)** Show that f has an inverse.

**(b)** Give a reason why the composite function $\mathrm{f}^{-1}\mathrm{g}$ exists. Find $\mathrm{f}^{-1}\mathrm{g}(x)$ and state the domain of $\mathrm{f}^{-1}\mathrm{g}$.

The function h is defined as follows.

$$\mathrm{h} : x \mapsto \ln(x - k + 1), \quad x \in \mathbb{R}, x \ge k, \text{ where } k < 0.$$

**(c)** Find the value of $k$ such that the range of hg is given by $(\ln 5, \infty)$.

## F Piecewise Functions

**21.** A curve has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \begin{cases} \frac{1}{3}x + \frac{2}{3} & \text{for } -2 \le x < 1, \\ \sqrt{5 - x} - 1 & \text{for } 1 \le x \le 4. \end{cases}$$

and that $\mathrm{f}(x) = \mathrm{f}(x + 6)$ for all real values of $x$.

**(a)** Sketch the graph of $y = \mathrm{f}(x)$ for $-5 \le x \le 11$.

**(b)** Evaluate $\mathrm{f}(-10) + \mathrm{f}(36)$.

**22.** The function f is defined by

$$\mathrm{f} : x \mapsto \begin{cases} |x| & \text{for } -2 < x < 0, \\ x + 2 & \text{for } 0 \le x < 2, \end{cases}$$

and that $\mathrm{f}(x) = \mathrm{f}(x + 4)$ for all real values of $x$.

**(a)** Sketch the graph of $y = \mathrm{f}(x)$ for $-4 < x < 3$.

**(b)** Show that $\mathrm{f}^{-1}$ does not exist.

**(c)** If the domain of f is restricted to $(a, 1]$, state the smallest value of $a$ such that $\mathrm{f}^{-1}$ exists.

**(d)** Using the domain found in part **(c)**, define $\mathrm{f}^{-1}$ in similar form.



**23.** The function f, with domain the set of non-negative integers, is given by

$$\mathrm{f}(n) \begin{cases} 1 & \text{for } n = 0, \\ 2\mathrm{f}\left(\frac{1}{2}n\right) & \text{for } n > 0, n \text{ even}, \\ 2 - \mathrm{f}(n - 1) & \text{for } n > 0, n \text{ odd}. \end{cases}$$

**(a)** Find $\mathrm{f}(3)$, $\mathrm{f}(4)$, and $\mathrm{f}(6)$.

**(b)** Does f have an inverse? Justify your answer.

**24.** The function f is defined by

$$\mathrm{f} : x \mapsto \mathrm{e}^{-(x+1)^2}, \text{ for } x \in \mathbb{R}, x > -1.$$

**(a)** Explain why $\mathrm{f}^{-1}$ exists, and define it in similar form.

**(b)** State the range of values of $x$ such that $\mathrm{ff}^{-1}(x) = x$.

The function g is defined by

$$\mathrm{g} : x \mapsto \begin{cases} 6 + 2x, & \text{if } -3 \le x \le -\frac{1}{2} \\ 4 - 2x, & \text{if } -\frac{1}{2} < x \le 3 \end{cases}$$

**(c)** Determine if the composite functions fg and gf exist.

**(d)** If the composite function exists, define it and find its range.

**25.** The function f is defined by

$$\mathrm{f}: x \mapsto \begin{cases} a\mathrm{e}^{a-x} & \text{for } 0 \le x < a, \\ \left|\dfrac{1}{a}(x-a)^2 - a\right| & \text{for } a \le x \le 2a. \end{cases}$$

**(a)** Define $\mathrm{f}^{-1}$ in similar form.

**(b)** Sketch the graph of $y = \mathrm{f}^{-1}(x)$, indicating clearly the axial intercepts.

The function g is defined by

$\mathrm{g}: x \mapsto \ln x \quad \text{for } x > 0.$

**(c)** Determine if $\mathrm{gf}^{-1}$ exists.

## G Applications

**26.** The researcher aims to understand how the number of work hours per day affects an individual's job satisfaction and, in turn, how job satisfaction influences the overall happiness of his family.The researcher developed two functions. The first function, f, relates the number of work hours, $x$ per day, measured in hours, to an individual's job satisfaction. It is defined as follows : $\mathrm{f}(x) = (x-5)^3 + 200,\ 0 \le x \le 15.$

**(a)** Is it necessary for a person to work long hours to have a high level of job satisfaction, according to function f ?

The second function, g, called the family happiness index, measures the overall happiness of the family based on an individual's satisfaction level ($w$) with their job. The index ranges from 0 to 1, with 0 representing the lowest level of happiness and 1 representing the highest level of happiness.

Function $w$ is defined by $\mathrm{g}(w) = \ln\left(\mathrm{e} - \dfrac{w}{1000}\right),\ 0 \le w \le 1000(\mathrm{e}-1).$



**(b)** Show that the composite function gf exists and interpret the meaning this function in this context.

**(c)** Find range of gf if an individual intends to work between 7 hours to 11 hours, inclusive.

**(d)** Determine whether gf increases or decreases as $x$ increases.

The researcher also developed a third model to examine how an individual personal savings in dollars relates to the number of work hours, $x$ per day. It is defined as follows : $\mathrm{h}(x) = 400 - (x-10)^2,\ 0 \le x \le 15,$ where $\mathrm{h}(x)$ denotes personal savings of an individual in dollars.

**(e)** Find the range of the personal savings in dollars, correct to nearest integer, if the happiness index based on individual's job satisfaction is at least 0.7 but not more than 0.9.

**27.** The outdoor temperature $\theta$, in degree Celsius, of a typical day in May, $t$ hours after midnight can be modelled by the function f such that $\mathrm{f}(t) = a\cos\left(\dfrac{\pi}{24}t - \dfrac{\pi}{2}\right) + b,\ 0 \le t < 24$ and $a, b > 0.$

It is given that on a typical day in May, the outdoor temperature is $25^\circ$C at midnight and the maximum temperature of the day is $38^\circ$C at noon.

**(a)** Find the value $b$ and show that $a = 13.$

**(ii)** It is given $\alpha$ and $\beta$, where $\alpha \ne \beta$, is such that $\mathrm{f}(\alpha) = \mathrm{f}(\beta).$

Show that $\alpha + \beta = k$, where $k$ is a constant to be determined.

The rate of absorption of energy, $\mathrm{g}(\theta)$ measured in kilowatt hours (kWh), of a solar panel system is dependent by the outdoor temperature $\theta$. It is defined as

$$\mathrm{g}: \theta \mapsto 50\ln\theta - 150, \theta > 23.$$

**(c)** Write, in context of the question, what the composite function gf represents and show that this function exists.

**(d)** Determine if gf has an inverse.

**(e)** Find the range of the rate of absorption of energy of a solar panel system occurs between $(12-s)$ am and $s$ pm on a typical day in May.

## H Mixed Exercise

**28.** The function f is defined by

$$\mathrm{f}: x \mapsto \sqrt{3}\sin x + \cos x, \quad x \in \mathbb{R}, \; -\pi < x < \frac{\pi}{6}.$$

**(a)** Express f in the form $R\sin(x+\alpha)$, where $R$ and $\alpha$ are exact constants to be determined, $R > 0$, $0 \le \alpha \le \frac{\pi}{2}$.

**(b)** Sketch f, giving the exact coordinates of the turning point and the end-points. Deduce the exact range of f.

**(c)** The domain of f is restricted such that the function $\mathrm{f}^{-1}$ exists. Find the largest domain of f for which $\mathrm{f}^{-1}$ exists. Define $\mathrm{f}^{-1}$ in a similar form.

**(d)** The function g is defined by

$$\mathrm{g}: x \mapsto \frac{1}{2} - |x-1|, x \in \mathbb{R}, \; -\frac{5}{2} \le x \le \frac{1}{2}$$

Explain why the composite function fg exists. Find the range of fg.

**29.** The function f is defined by

$$\mathrm{f}: x \to \begin{cases} a^2x - ax^2 & \text{for } x \le \frac{1}{2}a \\ -\frac{1}{2}a^2x + \frac{1}{2}a^3 & \text{for } x > \frac{1}{2}a, \end{cases}$$



where $a$ is a positive constant.

**(a)** Sketch the graph of $y = \mathrm{f}(x)$, indicating the coordinates of the points where the curve cuts the $x$-axis and other major features.

**(b)** If the domain of f is further restricted to $x \le k$, state with a reason the greatest value of $k$, in terms of $a$, for which the function $\mathrm{f}^{-1}$ exists.

**(c)** Using the value of $k$ found in part **(b)**, express the definition of $\mathrm{f}^{-1}$ in similar form. State the relationship between the graphs of f and $\mathrm{f}^{-1}$.

The function g is defined by

$$\mathrm{g}: x \to \mathrm{e}^x, \quad x \le a^3.$$

**(d)** State whether the composite function gf exists, justifying your answer.

**(e)** Find the range of gf.

**30.** The function f is defined by

$$\mathrm{f}: x \mapsto \frac{px}{x-1}, \quad \text{for } x \in \mathbb{R}, x \ne 1, \text{ where } p \text{ is a non-zero constant.}$$

It is given that f is self-inverse.

**(a)** Find the value of $p$.

Use the value of $p$ found in **(a)** for the rest of the question.

**(b)** Explain why $f^2$ exists.

**(c)** Find $f^2(x)$ and state its range.

**(d)** Hence, find $f^{2019}(x)$.

The functions g and h are defined by

$$\mathrm{g}: x \mapsto e^x + 2 \qquad \text{for } x \in \mathbb{R},$$
$$\mathrm{h}: x \mapsto \ln(ax^2 + bx + c) \quad \text{for } x \in \mathbb{R},$$

where $a, b$ and $c$ are constants.

**(e)** The graph of $y = \mathrm{h}(x)$ has a minimum point at $\left(\frac{1}{4}, \ln\frac{39}{8}\right)$. Given that $\mathrm{gh}(1) = 8$, find $\mathrm{h}(x)$.

**31.** The function f is defined by

$$\mathrm{f}(x) = \begin{cases} 4 - x & \text{for } 1 \le x < 3, \\ (x-4)^2 & \text{for } 3 \le x < 4, \end{cases}$$

and it is given that $\mathrm{f}(x-3) = \mathrm{f}(x)$ for all real values of $x$.

**(a)** State a reason why f does not have an inverse.

**(b)** Sketch the graph of $y = \mathrm{f}(x)$ for $-1 < x < 6$.

**(c)** Evaluate $\mathrm{f}(2017)$.

The function g has domain $[1, 4)$ and is defined by

$$\mathrm{g}(x) = \begin{cases} 4 - x & \text{for } 1 \le x < 3, \\ (x-4)^2 & \text{for } 3 \le x < 4, \end{cases}$$

**(d)** By sketching $y = \mathrm{g}(x)$ and $y = \mathrm{g}^{-1}(x)$ on the same diagram, state the values of $x$ such that $\mathrm{g}(x) = \mathrm{g}^{-1}(x)$.

The function h is defined by

$$\mathrm{h}(x) = \begin{cases} \sqrt{1-x} & \text{for } 0 \le x < 1, \\ (x-1) & \text{for } 1 \le x \le 3, \end{cases}$$

**(e)** Explain why $\mathrm{hg}^{-1}$ does not exist.

**(f)** Given that hg exists, define hg in similar form as function h.

**(g)** Find the range of hg.



## I Higher Order Thinking

**32.** Functions g and h are defined by

$$\mathrm{g}: x \mapsto \frac{a^2 x}{x - a}, \quad \text{for } x \in \mathbb{R},\ x \ne a, \text{ where } a > 1,$$
$$\mathrm{h}: x \mapsto x^2 \qquad \text{for } x \in \mathbb{R},\ x > a.$$

**(a)** Sketch the graph of g, indicating clearly the equations of the asymptotes in terms of $a$.

**(b)** Find the range of g and range of h in terms of $a$. Hence explain why $\mathrm{g}^{-1}\mathrm{h}$ exist.

**(c)** Find the range of $\mathrm{g}^{-1}\mathrm{h}$ in terms of $a$.

**(d)** Given that $\mathrm{g}^{-1}\mathrm{h}(m) = m$ where $m$ is an element of the domain of h, show that $m$ satisfies the equation $m^2 - am - a^2 = 0$. Find $m$ in terms of $a$.

**33.** The functions f and g are defined as follows:

$$\mathrm{f}: x \mapsto -\ln(x - b), \quad x > b \text{ and } b < 0,$$
$$\mathrm{g}: x \mapsto (x-a)^2 - 1, \quad x \in \mathbb{R}, \text{ where } a \text{ is a real constant.}$$

**(a)** Show that f is one-one and find $f^{-1}$ in similar form.

**(b)** Find the greatest integer value of $b$ such that the function fg exists.

**(c)** With the value of $b$ found in **(b)**, find the range of fg.

**34.** The functions f is defined by $f : x \mapsto \dfrac{1}{(x-2)^2}$, for $x < m$ where $m < 2$.

**(a)** Find the range of values of $m$ such that both $f^{-1}$ and $f^2$ exist.

**(b)** Given that the range of values of $m$ found in part **(a)** holds, find the functions $f^{-1}$ and $f^2$, and state their domains and ranges.

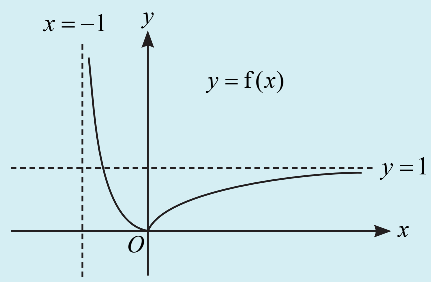
**35. (a)** The function g is defined by $g : x \mapsto x + \dfrac{\alpha}{x}$, $x \in \mathbb{R}$, $x > 0$.

**(i)** Determine the range of values of $\alpha$ such that $g^{-1}$ exists.

For the rest of the question, it is given that $\alpha = 4$ and the domain of g is further restricted to $x \geq \beta$.

**(ii)** Determine the least value of $\beta$ for which $g^{-1}$ exists.

Using this value of $\beta$, hence find $g^{-1}(x)$ and state its domain.




**(b)** The diagram shows the graph of function f defined on $(-1, \infty)$.

**(i)** State the restricted domain of f such that $f^{-1}$ exists and the range of f remains unchanged.

**(ii)** With the restricted domain of f found in part **(i)**, sketch the graphs of $f^{-1}$ and $ff^{-1}$ in a single diagram, showing clearly any asymptotes.

**(iii)** The function g and $f^{-1}g$ are given as follows:

$$g : x \mapsto 2 - \sqrt{1-x}, \quad -3 \leq x \leq 1.$$

$$f^{-1}g : x \mapsto \left(\frac{2-\sqrt{1-x}}{3-\sqrt{1-x}}\right), \quad -3 \leq x \leq 1.$$

Find $f^{-1}(x)$ and hence determine the rule of f.



**36.** The function f is defined by

$$f : x \mapsto e^{(2x+\alpha)^2}, x \in \mathbb{R}, x < k \text{ and } \alpha \text{ is a constant.}$$

**(a)** State the largest value of $k$ in terms of $\alpha$, for which the function $f^{-1}$ exists.

**(b)** Using the value of $k$ in **(a)**, find $f^{-1}(x)$ and write down the domain of $f^{-1}$.

In the rest of the questions, let $\alpha = 1$.

**(c)** Sketch on the same diagram, the graphs of $y = f^{-1}(x)$ and $y = f^{-1}f(x)$.

The function g is defined by

$$g : x \mapsto |x + a| + a, x \in \mathbb{R}, x < 0, a > 1.$$

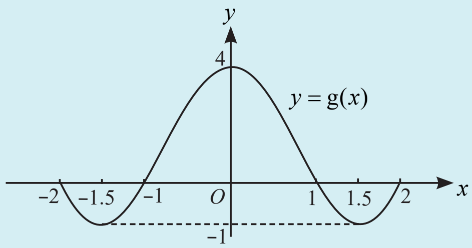
**(d)** On a separate diagram, sketch the graph of $y = g(x)$, indicating clearly the coordinates of any axial intercepts.

**(e)** Show that $gf^{-1}$ exists and find the exact range of $gf^{-1}$.

**37.** The function f is defined by

$$f : x \mapsto |x| - k, -2k < x < 2k, \text{ where } k > 2.$$

The diagram shows the graph of the function, $y = g(x)$, where $-2 \leq x \leq 2$. The graph crosses the $x$-axis at $x = -2$, $x = -1$, $x = 1$ and $x = 2$, and has turning points at $(-1.5, -1)$, $(0, 4)$ and $(1.5, -1)$.



**(a)** Explain why the composite function fg exists.

**(b)** Find in terms of $k$,

**(i)** the value of $\text{fg}(-1)$,

**(ii)** the range of fg.

**(c)** Given that $k = 3$, sketch the graph of $y = \text{fg}(x)$, stating the coordinates of the turning points, if any.

## J Examination Style Questions

**38.** The function f is defined by $\text{f} : x \mapsto 1 + \dfrac{5}{x^2 + 2x + 3}$, $x \geq 0$.

**(a)** Sketch the graph of f, indicating clearly the equation of any asymptotes and the exact coordinates of any point(s) of intersection with the axes.

**(b)** Show that $\text{f}^{-1}$ exists and find $\text{f}^{-1}(x)$, stating the exact domain of $\text{f}^{-1}$.

**(c)** On the same diagram as in part **(a)**, sketch the graphs of $y = \text{f}^{-1}(x)$ and $y = \text{ff}^{-1}(x)$.

**(d)** Show that $\text{f}^2$ exists and find the exact range of $\text{f}^2$.

**39.** The function f is defined by

$$\text{f} : x \mapsto x^2 + \lambda x - \lambda^2, \quad x \in \mathbb{R}, \; x \leq -\frac{\lambda}{2}, \text{ where } \lambda \text{ is a positive real number.}$$

**(a)** Find $\text{f}^{-1}(x)$ and write down the domain and range of $\text{f}^{-1}$.

The function g is defined by

$$\text{g} : x \mapsto \text{e}^x, \quad x \in \mathbb{R}.$$

**(b)** Explain why the composite function fg does not exist.

**(c)** Find the composite function gf in a similar form and find its range.

**40.** Functions f and g are defined by

$$\text{f} : x \mapsto \frac{x+a}{x+b}, \quad \text{for } x \in \mathbb{R}, x \neq -b, a \neq -1,$$

$$\text{g} : x \mapsto x, \quad \text{for } x \in \mathbb{R}.$$

It is given that ff = g.

**(a)** Find the value of $b$.

**(b)** Find $\text{f}^{-1}(x)$ in terms of $x$ and $a$.

**(c)** Establish a relationship such that $\text{f}(x)$ is not a self-inverse.

**41.** Functions f and g are defined by

$$\text{f} : x \mapsto -2x^2 + 4x, \quad x \in \{-2, -1, 0, 1, 2\}.$$

$$\text{g} : x \mapsto \frac{ax}{x-a}, \quad x \in \mathbb{R}, x \neq a, \text{ where } a \text{ is a positive constant.}$$

**(a)** Find the range of f and explain whether f has an inverse.

**(b)** State the value of $a$ such that gf does not exist.

**(c)** Show that $\text{g}(x) = \text{g}^{-1}(x)$ for all $x$, $x \neq a$.

It is given that $a = 3$.

**(d)** Hence, or otherwise, find the value $\text{g}^{2019}\text{f}(1)$.

**42.** **(a)** A function f is said to be self-inverse if $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ for all $x$ in the domain of f.

The function f is defined by

$$\mathrm{f} : x \mapsto \frac{3x+k}{x-b}, \quad x \in \mathbb{R},\ x \neq b, \text{ where } k \text{ and } b \text{ are constants.}$$

**(i)** Find the value of $b$ and the set of values of $k$ such that f is self-inverse.

Another function g is defined by

$$\mathrm{fg} : x \mapsto 2x-1, \quad x \neq 2.$$

**(ii)** Using the value of $b$ found in **(i)**, find in terms of $k$, an expression for $\mathrm{g}(x)$.

**(b)** The function h is defined as follows:

$$\mathrm{h}(x) = \begin{cases} -4x+8, & \text{for } 1 \le x < 2, \\ -x^2+8x-12 & \text{for } 2 \le x < 4. \end{cases}$$

and that $\mathrm{h}(x+3) = \mathrm{h}(x)$ for all real values of $x$.

Sketch the graph of $y = \mathrm{h}(x)$ for $-4 \le x \le 6$, indicating the axial intercepts and endpoints clearly.

**43.** The functions f and g are defined by

$$\mathrm{f} : x \mapsto (x-3)^4, \quad \text{for } x \in \mathbb{R},\ x \ge a,$$
$$\mathrm{g} : x \mapsto \ln(x-3), \quad \text{for } x \in \mathbb{R},\ x > 3,$$

where $a$ is a constant.

**(a)** Determine whether f has an inverse when $a = 1$.

**(b)** It is given that $\mathrm{f}^{-1}$ exists. State the minimum value of $a$, and define $\mathrm{f}^{-1}$ in a similar form with this minimum value of $a$.

For the rest of the question, let $a = 5$.

**(c)** Show that the composite function gf exists. Hence solve $\mathrm{gf}(x) = \ln 78$.

**44.** The function f is defined by

$$\mathrm{f} : x \mapsto 1 + 2e^{-x^2}, \ x \in \mathbb{R}.$$

**(a)** Show that f does not have an inverse.

**(b)** The domain of f is further restricted to $x \le k$, state the largest value of $k$ for which the function $\mathrm{f}^{-1}$ exists.

In the rest of the question, the domain of f is $x \in \mathbb{R},\ x \le k$, with the value of $k$ found in part **(b)**.

**(c)** Find $\mathrm{f}^{-1}(x)$.

The function g has an inverse such that the range of $\mathrm{g}^{-1}$ is given by $(1, 3]$.

**(d)** Explain why the composite function gf exists.

It is given that the composite function gf is defined by $\mathrm{gf}(x) = x$.

**(e)** State the domain and range of gf.

**(f)** By considering $\mathrm{gf}(-2)$, find the exact value of $\mathrm{g}^{-1}(-2)$.

**45.** The function f is defined by

$$\mathrm{f} : x \mapsto x^2 + 4x - 5, \quad \text{for } x \le k,\ k \in \mathbb{R}.$$

**(a)** Find the largest exact value of $k$ such that $\mathrm{f}^{-1}$ exists. For this value of $k$, define $\mathrm{f}^{-1}$ in a similar form.

Another function g is defined by

$$\mathrm{g} : x \mapsto \begin{cases} 4-x^2, & \text{for } 0 < x \le 2 \\ 2x-4, & \text{for } 2 < x \le 4 \end{cases}$$

and that $\mathrm{g}(x) = \mathrm{g}(x+4)$ for all real values of $x$.

**(b)** Sketch the graph of $y = \mathrm{g}(x)$ for $-1 < x \le 7$.

**(c)** Using the results in part **(a)** and **(b)** explain why composite function $\mathrm{f}^{-1}\mathrm{g}$ exists and find the exact value of $\mathrm{f}^{-1}\mathrm{g}(6)$.